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# МАЗМҰНЫ — СОДЕРЖАНИЕ — CONTENTS

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# A nonlocal problem for loaded partial differential equations of fourth order

A nonlocal problem for the fourth order system of loaded partial differential equations is considered. The questions of a existence unique solution of the considered problem and ways of its construction are investigated. The nonlocal problem for the loaded partial differential equation of fourth order is reduced to a nonlocal problem for a system of loaded hyperbolic equations of second order with integral conditions by introducing new functions. As a result of solving nonlocal problem with integral conditions is applied a method of introduction functional parameters. The algorithms of finding the approximate solution to the nonlocal problem with integral conditions for the system of loaded hyperbolic equations are proposed and their convergence is proved. The conditions of the unique solvability of the nonlocal problem for the loaded hyperbolic equations are obtained in the terms of initial data. The results also formulated relative to the original problem.

*Keywords:* nonlocal problem, loaded partial differential equations of fourth order, integral condition, system of loaded hyperbolic equations, algorithm, unique solvability.

## *Introduction*

Many problems of dynamics and kinetics of gas sorption, processes of drying by air stream, movement of adsorbed mixtures and others lead to the study of nonlocal problems for the systems of hyperbolic equations with loading [1–10] and also for nonlocal problems with integral conditions for equations of hyperbolic type [11–16]. In order to solve these problems, the theoretical methods of ordinary differential equations, loaded differential equations, numerical-analytical method are applied, and new approaches and methods are developed as well. Conditions for solvability are received and ways for finding the approximate solutions are offered. Mathematical modeling of various processes in physics, chemistry, biology, technology, ecology, economics and others are leaded to nonlocal problems for the higher order loaded differential equations with variable coefficients and parameters. Despite the presence of numerous works, general statements of nonlocal problems for the higher order loaded partial differential equations remain poorly studied up to now. Therefore, the problems of solvability of nonlocal problems for the fourth order partial differential equations with and without loading remain important for applications [17–21].

The Goal of this paper is to study boundary value problems with data on the characteristics for the fourth order system of hyperbolic equations with loading and to establish coefficient criteria for unique solubility and to construct algorithms for finding their approximate solutions. Therefore, in the present paper we study of a questions the existence and uniqueness of classical solutions to nonlocal problem for the fourth order system of loaded partial differential equations and the methods of finding its approximate solutions. For these purposes, we are applied method of introduction a new functions [22, 23] for solve of this problem.

We consider on the domain $\Omega = [0,T] \times [0,\omega]$ a nonlocal problem for the fourth order system of the loaded partial differential equations with two independent variables

$$\frac{\partial^4 u}{\partial x^3 \partial t} = \sum_{i=1}^{3} \Big\{ A_i(t,x) \frac{\partial^{4-i} u}{\partial x^{4-i}} + B_i(t,x) \frac{\partial^{4-i} u}{\partial x^{3-i} \partial t} \Big\} + C(t,x)u +$$

$$+ \sum_{i=1}^{3} \sum_{k=1}^{m} \Big\{ K_{i,k}(t,x) \frac{\partial^{4-i} u(t,x)}{\partial x^{4-i}} + L_{i,k}(t,x) \frac{\partial^{4-i} u(t,x)}{\partial x^{3-i} \partial t} \Big\}\Big|_{t=t_k} + \sum_{k=1}^{m} M_{i,k}(t,x) u(t_k,x) + f(t,x), \qquad (1)$$

$$P(x) \frac{\partial^3 u(0,x)}{\partial x^3} + S(x) \frac{\partial^3 u(T,x)}{\partial x^3} = \varphi(x), \qquad x \in [0,\omega], \qquad (2)$$

$$u(t,0) = \psi_0(t), \qquad \frac{\partial u(t,x)}{\partial x}\Big|_{x=0} = \psi_1(t), \qquad \frac{\partial^2 u(t,x)}{\partial x^2}\Big|_{x=0} = \psi_2(t), \qquad t \in [0,T]. \qquad (3)$$

Here $u(t,x) = col(u_1(t,x), u_2(t,x), ..., u_n(t,x))$ is unknown function; the $n \times n$ matrices $A_i(t,x)$, $B_i(t,x)$, $C(t,x)$, $K_{i,k}(t,x)$, $L_{i,k}(t,x)$, $M_{i,k}(t,x)$, $i = \overline{1,3}$, $k = \overline{1,m}$, and $n$ vector function $f(t,x)$ are continuous on $\Omega$; $0 \le t_1 < t_2 < ... < t_m \le T$; the $n \times n$ matrices $P(x)$, $S(x)$, and $n$ vector function $\varphi(x)$ are continuous on $[0,\omega]$; the $n$ vector-functions $\psi_0(t)$, $\psi_1(t)$ and $\psi_2(t)$ are continuously differentiable on $[0,T]$.

Let $C(\Omega, R^n)$ be the space of continuous vector functions $u : \Omega \to R^n$ on $\Omega$ with norm $||u||_0 = \max_{(t,x)\in\Omega} ||u(t,x)||$.

A function $u(t,x) \in C(\Omega, R^n)$, having partial derivatives $\dfrac{\partial^{i+j} u(t,x)}{\partial x^i \partial t^j} \in C(\Omega, R^n)$, $i = \overline{1,3}$, $j = 0,1$, is called a *classical solution* to problem (1)–(3) if it satisfies to system of loaded equations (1) for all $(t,x) \in \Omega$ and meets the conditions (2) and (3).

We will investigate the existence of a unique solution to the nonlocal problem for the fourth order loaded partial differential equation (1)–(3). We use method of introduction a new functions for solve of the problem (1)–(3) and construct of its approximate solutions. The nonlocal problem for the fourth order system of loaded partial differential equations is reduced to a nonlocal problem for a system of loaded hyperbolic equations of second order with integral conditions by introducing new functions. An algorithms of finding the approximate solution to the equivalent nonlocal problem with integral conditions are constructed and their convergence is proved. The conditions of the unique solvability of the nonlocal problem for the system of loaded hyperbolic equations with integral conditions are established in the terms of initial data. The results also formulated relative to the original of the nonlocal problem for the fourth order system of loaded partial differential equations.

## *1 Scheme of the method*

Introduce a new unknown functions $\quad w(t,x) = \dfrac{\partial^2 u(t,x)}{\partial x^2}, \quad v(t,x) = \dfrac{\partial u(t,x)}{\partial x}.$

Taking into account of first and second conditions in (3), we have

$$v(t,x) = \psi_1(t) + \int_0^x w(t,\xi) d\xi, \qquad u(t,x) = \psi_0(t) + \psi_1(t)x + \int_0^x \int_0^\xi w(t,\xi_1) d\xi_1 d\xi.$$

Then problem (1)–(3) is reduced to a following problem

$$\frac{\partial^2 w}{\partial x \partial t} = A_1(t,x) \frac{\partial w}{\partial x} + B_1(t,x) \frac{\partial w}{\partial t} + A_2(t,x) w +$$

$$+ \sum_{k=1}^{m} \Big\{ K_{1,k}(t,x) \frac{\partial w(t_k,x)}{\partial x} + L_{1,k}(t,x) \frac{\partial w(t,x)}{\partial t}\Big|_{t=t_k} + K_{2,k}(t,x) w(t_k,x) \Big\} + f(t,x) + g(t,x,v,u), \qquad (4)$$

$$P(x) \frac{\partial w(0,x)}{\partial x} + S(x) \frac{\partial w(T,x)}{\partial x} = \varphi(x), \qquad (5)$$

$$w(t,0)=\psi_2(t), \qquad t\in[0,T], \tag{6}$$

$$v(t,x)=\psi_1(t)+\int_0^x w(t,\xi)d\xi, \qquad u(t,x)=\psi_0(t)+\psi_1(t)x+\int_0^x\int_0^\xi w(t,\xi_1)d\xi_1 d\xi, \tag{7}$$

where

$$g(t,x,v,u)=A_3(t,x)v(t,x)+B_2(t,x)\frac{\partial v}{\partial t}+B_3(t,x)\frac{\partial u}{\partial t}+C(t,x)u+$$

$$+\sum_{k=1}^m\Big\{K_{3,k}(t,x)v(t_k,x)+L_{2,k}(t,x)\frac{\partial v(t,x)}{\partial t}\Big|_{t=t_k}+L_{3,k}(t,x)\frac{\partial u(t,x)}{\partial t}\Big|_{t=t_k}+M_{i,k}(t,x)u(t_k,x)\Big\}.$$

From (7) it follows

$$\frac{\partial v(t,x)}{\partial t}=\dot\psi_1(t)+\int_0^x\frac{\partial w(t,\xi)}{\partial t}d\xi, \qquad \frac{\partial u(t,x)}{\partial t}=\dot\psi_0(t)+\dot\psi_1(t)x+\int_0^x\int_0^\xi\frac{\partial w(t,\xi_1)}{\partial t}d\xi_1 d\xi. \tag{8}$$

A triple functions $(w(t,x),v(t,x),u(t,x))$, where $w(t,x)\in C(\Omega,\mathbb{R}^n)$, $\frac{\partial w(t,x)}{\partial x}\in C(\Omega,\mathbb{R}^n)$, $\frac{\partial w(t,x)}{\partial t}\in C(\Omega,\mathbb{R}^n)$, $\frac{\partial^2 w(t,x)}{\partial x\partial t}\in C(\Omega,\mathbb{R}^n)$, and $v(t,x)\in C(\Omega,\mathbb{R}^n)$, $\frac{\partial v(t,x)}{\partial t}\in C(\Omega,\mathbb{R}^n)$, $u(t,x)\in C(\Omega,\mathbb{R}^n)$, $\frac{\partial u(t,x)}{\partial t}\in C(\Omega,\mathbb{R}^n)$, is called a solution to problem (4)–(7), if it satisfies the system of loaded hyperbolic equations second order (4) for all $(t,x)\in\Omega$, the boundary conditions (5), (6), and integral relations (7).

The problem (4)–(6) at fixed $v(t,x)$, $u(t,x)$, is a nonlocal problem for system of loaded hyperbolic equations of second order with respect to $w(t,x)$ on $\Omega$. The integral relations (7) allow us to determine the unknown functions $v(t,x)$ and $u(t,x)$.

From (8) we define the partial derivatives $\frac{\partial v(t,x)}{\partial t}$ and $\frac{\partial u(t,x)}{\partial t}$ for all $(t,x)\in\Omega$.

The problem (4)–(6) can be interpreted:

- as a nonlocal problem for the system of loaded hyperbolic equations of second order with distributed parameters $v(t,x)$ and $u(t,x)$;
- as an inverse problem for the system of loaded hyperbolic equations of second order, where the unknown functions $v(t,x)$, $u(t,x)$ determine from integral relations (7);
- as a control problem for the system of loaded hyperbolic equations of second order, where the control functions $v(t,x)$, $u(t,x)$ satisfy integral constrains (7).

Since the function $w(t,x)$ and the functions $v(t,x)$, $u(t,x)$ are unknown together to find a solution to problem (4)–(7) we use an iterative method.

### *2 Algorithm for finding of solution to problem (4)–(7)*

A triple functions $(w^*(t,x),v^*(t,x),u^*(t,x))$ we determine as a limit of sequences of triple functions $(w^{(p)}(t,x)$, $v^{(p)}(t,x)$, $u^{(p)}(t,x))$ and $p=0,1,2,...$, by the following algorithm:

*Step - 0.* 1) Let $v(t,x)=\psi_1(t)$, $u(t,x)=\psi_0(t)+\psi_1(t)x$, $\frac{\partial v(t,x)}{\partial t}=\dot\psi_1(t)$, $\frac{\partial u(t,x)}{\partial t}=\dot\psi_0(t)+\dot\psi_1(t)x$ in right-hand side of system (4). Then from nonlocal problem for the system of loaded hyperbolic equations (4)–(6) we find $w^{(0)}(t,x)$ for all $(t,x)\in\Omega$. Also we find its partial derivatives $\frac{\partial w^{(0)}(t,x)}{\partial x}$, $\frac{\partial w^{(0)}(t,x)}{\partial t}$ and $\frac{\partial^2 w^{(0)}(t,x)}{\partial x\partial t}$ for all $(t,x)\in\Omega$;

2) From integral relations (7) we determine $v^{(0)}(t,x)$ and $u^{(0)}(t,x)$:

$$v^{(0)}(t,x)=\psi_1(t)+\int_0^x w^{(0)}(t,\xi)d\xi, \quad u^{(0)}(t,x)=\psi_0(t)+\psi_1(t)x+\int_0^x\int_0^\xi w^{(0)}(t,\xi_1)d\xi_1 d\xi, \qquad (t,x)\in\Omega.$$

Then from (8) we find $\frac{\partial v^{(0)}(t,x)}{\partial t}$ and $\frac{\partial u^{(0)}(t,x)}{\partial t}$:

$$\frac{\partial v^{(0)}(t,x)}{\partial t}=\dot\psi_1(t)+\int_0^x\frac{\partial w^{(0)}(t,\xi)}{\partial t}d\xi, \quad \frac{\partial u^{(0)}(t,x)}{\partial t}=\dot\psi_0(t)+\dot\psi_1(t)+\int_0^x\int_0^\xi\frac{\partial w^{(0)}(t,\xi_1)}{\partial t}d\xi_1 d\xi,$$

And so on.

*Step - p.* 1) Suppose that $v(t,x) = v^{(p-1)}(t,x)$, $u(t,x) = u^{(p-1)}(t,x)$, $\dfrac{\partial v(t,x)}{\partial t} = \dfrac{\partial v^{(p-1)}(t,x)}{\partial t}$ and $\dfrac{\partial u(t,x)}{\partial t} = = \dfrac{\partial u^{(p-1)}(t,x)}{\partial t}$ in right-hand side of system (4). Then from nonlocal problem for the system of hyperbolic equations (4)–(6) we find $w^{(p)}(t,x)$ for all $(t,x) \in \Omega$. Also we find its partial derivatives $\dfrac{\partial w^{(p)}(t,x)}{\partial x}$, $\dfrac{\partial w^{(p)}(t,x)}{\partial t}$ and $\dfrac{\partial^2 w^{(p)}(t,x)}{\partial x \partial t}$ for all $(t,x) \in \Omega$.

2) From integral relations (7) we determine $v^{(p)}(t,x)$ and $u^{(p)}(t,x)$:

$$v^{(p)}(t,x) = \psi_1(t) + \int_0^x w^{(p)}(t,\xi)d\xi, \quad u^{(p)}(t,x) = \psi_0(t) + \psi_1(t)x + \int_0^x \int_0^\xi w^{(p)}(t,\xi_1)d\xi_1 d\xi, \qquad (t,x) \in \Omega.$$

Then from (8) we find $\dfrac{\partial v^{(p)}(t,x)}{\partial t}$ and $\dfrac{\partial u^{(p)}(t,x)}{\partial t}$:

$$\frac{\partial v^{(p)}(t,x)}{\partial t} = \dot{\psi}_1(t) + \int_0^x \frac{\partial w^{(p)}(t,\xi)}{\partial t} d\xi, \quad \frac{\partial u^{(p)}(t,x)}{\partial t} = \dot{\psi}_0(t) + \dot{\psi}_1(t) + \int_0^x \int_0^\xi \frac{\partial w^{(p)}(t,\xi_1)}{\partial t} d\xi_1 d\xi,$$

$p = 1, 2, \ldots.$

### *3 Nonlocal problem for system of loaded hyperbolic equations*

We also consider an auxiliary nonlocal problem for system of loaded hyperbolic equations second order

$$\frac{\partial^2 w}{\partial x \partial t} = A_1(t,x)\frac{\partial w}{\partial x} + B_1(t,x)\frac{\partial w}{\partial t} + A_2(t,x)w +$$

$$+ \sum_{k=1}^m \Big\{ K_{1,k}(t,x)\frac{\partial w(t_k,x)}{\partial x} + L_{1,k}(t,x)\frac{\partial w(t,x)}{\partial t}\Big|_{t=t_k} + K_{2,k}(t,x)w(t_k,x)\Big\} + F(t,x), \tag{9}$$

$$P(x)\frac{\partial w(0,x)}{\partial x} + S(x)\frac{\partial w(T,x)}{\partial x} = \varphi(x), \qquad x \in [0,\omega], \tag{10}$$

$$w(t,0) = \psi_2(t), \qquad t \in [0,T]. \tag{11}$$

Here the functions $F(t,x) \in C(\Omega, \mathbb{R}^n)$.

Let $t_0 = 0$, $t_{m+1} = T$.

By lines of loading $t = t_k$, $k = \overline{1,m}$, we divide of domain $\Omega = \bigcup_{r=1}^{m+1} \Omega_r$, where $\Omega_r = [t_{r-1}, t_r] \times [0,\omega]$, $r = \overline{1,m+1}$. By $w_r(t,x)$ denote the restriction of function $w(t,x)$ to the subdomain $\Omega_r$ such that $w_r : \Omega_r \to R^n$ and $w_r(t,x) = w(t,x)$ for all $(t,x) \in \Omega_r$ and $r = \overline{1,m+1}$.

Further, by $\lambda_r(x)$ denote the value of $w_r(t,x)$ under $t = t_{r-1}$, $r = \overline{1,m+1}$. We replace $w_r(t,x)$ by $\widetilde{w}_r(t,x) + \lambda_r(x)$ in each domain $\Omega_r$, $r = \overline{1,m+1}$. This implies $\widetilde{w}_r(t_{r-1},x) = 0$, and $\dfrac{\partial \widetilde{w}_r(t_{r-1},x)}{\partial x} = 0$, for all $x \in [0,\omega]$ and $r = \overline{1,m+1}$.

Then the problem (9)–(11) is equivalent to the problem with unknown functions $\lambda_r(x)$:

$$\frac{\partial^2 \widetilde{w}_r}{\partial x \partial t} = A_1(t,x)\frac{\partial \widetilde{w}_r}{\partial x} + A_1(t,x)\dot{\lambda}_r(x) + B_1(t,x)\frac{\partial \widetilde{w}_r}{\partial t} + A_2(t,x)\widetilde{w}_r + A_2(t,x)\lambda_r(x) +$$

$$+ \sum_{k=1}^m \Big\{ K_{1,k}(t,x)\dot{\lambda}_{k+1}(x) + K_{2,k}(t,x)\lambda_{k+1}(x)\Big\} + \sum_{k=1}^m L_{1,k}(t,x)\frac{\partial \widetilde{w}_{k+1}(t,x)}{\partial t}\Big|_{t=t_k} + F(t,x), \tag{12}$$

$$\widetilde{w}_r(t_{r-1},x) = 0, \qquad x \in [0,\omega], \qquad r = \overline{1,m+1}, \tag{13}$$

$$\widetilde{w}_r(t,0) = \psi_2(t) - \psi_2(t_{r-1}), \qquad t \in [t_{r-1}, t_r], \qquad r = \overline{1,m+1}, \tag{14}$$

$$P(x)\dot{\lambda}_1(x)+S(x)\dot{\lambda}_{m+1}(x)+S(x)\frac{\partial\widetilde{w}_{m+1}(t_{m+1},x)}{\partial x}=\varphi(x),\qquad x\in[0,\omega],\tag{15}$$

$$\frac{\partial\widetilde{w}_s(t_s,x)}{\partial x}+\dot{\lambda}_s(x)=\dot{\lambda}_{s+1}(x),\qquad x\in[0,\omega],\qquad s=\overline{1,m}.\tag{16}$$

Here relations (16) are conditions of continuity at interior lines $t=t_s$, $s=\overline{1,m}$ of desired function $w(t,x)$.

The problems (9)–(11) and (12)–(16) are equivalent in the following sense. If the function $w(t,x)$ is a classical solution to (9)–(11), then system of pairs $(\lambda_r(x),\widetilde{w}_r(t,x))$, where $\lambda_r(x)=w(t_{r-1},x)$ and $\widetilde{w}_r(t,x)=w(t,x)-w(t_{r-1},x)$, and $(t,x)\in\Omega_r$, and $r=\overline{1,m+1}$ is a solution to problem (12)–(16). Conversely, if the system of pairs $(\lambda_r^*(x),\widetilde{w}_r^*(t,x))$, $(t,x)\in\Omega_r$, and $r=\overline{1,m+1}$, is a solution to (12)–(16), then the function $w^*(t,x)$ defined by the equalities

$$w^*(t,x)=\lambda_r^*(x)+\widetilde{w}_r^*(t,x)\quad\text{for all}\quad(t,x)\in\Omega_r,\quad\text{and}\quad r=\overline{1,m+1},$$

is a classical solution to problem (9)–(11).

From compatibility condition at $(0,0)$ we obtain:

$$\lambda_r(0)=\psi_2(t_{r-1}),\qquad r=\overline{1,m+1}.\tag{17}$$

At fixed $\lambda_r$ problem (12)–(14) is Goursat problem for system of loaded hyperbolic equations of second order on $\Omega_r$ with respect to $\widetilde{w}_r(t,x)$, $r=\overline{1,m+1}$.

Let $\widetilde{V}_r(t,x)=\frac{\partial\widetilde{w}_r(t,x)}{\partial x}$, $\widetilde{W}_r(t,x)=\frac{\partial\widetilde{w}_r(t,x)}{\partial t}$.

Goursat problem (12)–(14) is equivalent to the system of three integral equations on $\Omega_r$ at fixed $\lambda_r(x)$

$$\widetilde{V}_r(t,x)=\int_{t_{r-1}}^{t}\Big\{A_1(\tau,x)\widetilde{V}_r(\tau,x)+B_1(\tau,x)\widetilde{W}_r(\tau,x)+A_2(\tau,x)\widetilde{w}_r(\tau,x)+\sum_{k=1}^{m}L_{1,k}(\tau,x)\widetilde{W}_{k+1}(t_k,x)+$$

$$+F(\tau,x)+A_1(\tau,x)\dot{\lambda}_r(x)+A_2(\tau,x)\lambda_r(x)+\sum_{k=1}^{m}\Big\{K_{1,k}(\tau,x)\dot{\lambda}_{k+1}(x)+K_{2,k}(\tau,x)\lambda_{k+1}(x)\Big\}\Big\}d\tau,\tag{18}$$

$$\widetilde{W}_r(t,x)=\dot{\psi}_2(t)+\int_{0}^{x}\Big\{A_1(t,\xi)\widetilde{V}_r(t,\xi)+B_1(t,\xi)\widetilde{W}_r(t,\xi)+A_2(t,\xi)\widetilde{w}_r(t,\xi)+\sum_{k=1}^{m}L_{1,k}(t,\xi)\widetilde{W}_{k+1}(t_k,\xi)+$$

$$+F(t,\xi)+A_1(t,\xi)\dot{\lambda}_r(\xi)+A_2(t,\xi)\lambda_r(\xi)+\sum_{k=1}^{m}\Big\{K_{1,k}(t,\xi)\dot{\lambda}_{k+1}(\xi)+K_{2,k}(t,\xi)\lambda_{k+1}(\xi)\Big\}\Big\}d\xi,\tag{19}$$

$$\widetilde{w}_r(t,x)=\psi_2(t)-\psi_2(t_{r-1})+\int_{t_{r-1}}^{t}\widetilde{W}_r(\tau,x)d\tau.\tag{20}$$

Substituting $\widetilde{V}_r(\tau,x)=\frac{\partial\widetilde{w}_r(\tau,x)}{\partial x}$ in the right-hand side (18) and repeating the process $\nu$ times, and $\nu\in\mathbb{N}$, we obtain

$$\widetilde{V}_r(t,x)=D_{\nu,r}(t,x)\dot{\lambda}_r(x)+\sum_{k=1}^{m}\widetilde{D}_{\nu,r,k}(t,x)\dot{\lambda}_{k+1}(x)+E_{\nu,r}(t,x)\lambda_r(x)+\sum_{k=1}^{m}\widetilde{E}_{\nu,r,k}(t,x)\lambda_{k+1}(x)+$$

$$+G_{\nu,r}(t,x,\widetilde{V}_r)+H_{\nu,r}(t,x,\widetilde{W}_r,\widetilde{w}_r)+F_{\nu,r}(t,x),\tag{21}$$

where

$$D_{\nu,r}(t,x)=\int_{t_{r-1}}^{t}A_1(\tau,x)d\tau+\int_{t_{r-1}}^{t}A_1(\tau_1,x)\int_{t_{r-1}}^{\tau_1}A_1(\tau_2,x)d\tau_2d\tau_1+...+$$

$$+\int_{t_{r-1}}^{t}A_1(\tau_1,x)\int_{t_{r-1}}^{\tau_1}A_1(\tau_2,x)...\int_{t_{r-1}}^{\tau_{\nu-1}}A_1(\tau_\nu,x)d\tau_\nu d\tau_{\nu-1}...d\tau_2d\tau_1,$$

$$\widetilde{D}_{\nu,r,k}(t,x)=\int_{t_{r-1}}^{t}K_{1,k}(\tau,x)d\tau+\int_{t_{r-1}}^{t}A_1(\tau,x)\int_{t_{r-1}}^{\tau}K_{1,k}(\tau_1,x)d\tau_1d\tau+$$

$$+\int_{t_{r-1}}^{t}A_1(\tau,x)\int_{t_{r-1}}^{\tau}A_1(\tau_1,x)\int_{t_{r-1}}^{\tau_1}K_{1,k}(\tau_2,x)d\tau_2d\tau_1d\tau+...+$$

$$+\int_{t_{r-1}}^{t} A_1(\tau_1,x)...\int_{t_{r-1}}^{\tau_{\nu-1}} A_1(\tau_\nu,x)\int_{t_{r-1}}^{\tau_\nu} K_{1,k}(\tau_{\nu+1},x)d\tau_{\nu+1}d\tau_\nu...d\tau_1,$$

$$E_{\nu,r}(t,x)=\int_{t_{r-1}}^{t} A_2(\tau,x)d\tau+\int_{t_{r-1}}^{\tau} A_1(\tau,x)\int_{t_{r-1}}^{\tau_1} A_2(\tau,x)d\tau_1 d\tau+$$

$$+\int_{t_{r-1}}^{t} A_1(\tau,x)\int_{t_{r-1}}^{\tau} A_1(\tau_1,x)\int_{t_{r-1}}^{\tau_1} A_2(\tau_2,x)d\tau_2 d\tau_1 d\tau+...+$$

$$+\int_{t_{r-1}}^{t} A_1(\tau_1,x)...\int_{t_{r-1}}^{\tau_{\nu-1}} A_1(\tau_\nu,x)\int_{t_{r-1}}^{\tau_\nu} A_2(\tau_{\nu+1},x)d\tau_{\nu+1}d\tau_\nu...d\tau_1,$$

$$\widetilde{E}_{\nu,r,k}(t,x)=\int_{t_{r-1}}^{t} K_{2,k}(\tau,x)d\tau+\int_{t_{r-1}}^{t} A_1(\tau,x)\int_{t_{r-1}}^{\tau} K_{2,k}(\tau_1,x)d\tau_1 d\tau+$$

$$+\int_{t_{r-1}}^{t} A_1(\tau,x)\int_{t_{r-1}}^{\tau} A_1(\tau_1,x)\int_{t_{r-1}}^{\tau_1} K_{2,k}(\tau_2,x)d\tau_2 d\tau_1 d\tau+...+$$

$$+\int_{t_{r-1}}^{t} A_1(\tau_1,x)...\int_{t_{r-1}}^{\tau_{\nu-1}} A_1(\tau_\nu,x)\int_{t_{r-1}}^{\tau_\nu} K_{2,k}(\tau_{\nu+1},x)d\tau_{\nu+1}d\tau_\nu...d\tau_1,$$

$$G_{\nu,r}(t,x,\widetilde{V}_r)=\int_{t_{r-1}}^{t} A_1(\tau_1,x)...\int_{t_{r-1}}^{\tau_{\nu-1}} A_1(\tau_\nu,x)\widetilde{V}_r(\tau_\nu,x)d\tau_\nu...d\tau_2 d\tau_1,$$

$$H_{\nu,r}(t,x,\widetilde{W}_r,\widetilde{w}_r)=\int_{t_{r-1}}^{t}\Big[B_1(\tau,x)\widetilde{W}_r(\tau,x)+A_2(\tau,x)\widetilde{w}_r(\tau,x)+\sum_{k=1}^{m} L_{1,k}(\tau,x)\widetilde{W}_{k+1}(t_k,x)\Big]d\tau+$$

$$+\int_{t_{r-1}}^{t} A_1(\tau_1,x)\int_{t_{r-1}}^{\tau_1}\Big[B_1(\tau_2,x)\widetilde{W}_r(\tau_2,x)+A_2(\tau_2,x)\widetilde{w}_r(\tau_2,x)+\sum_{k=1}^{m} L_{1,k}(\tau_2,x)\widetilde{W}_{k+1}(t_k,x)\Big]d\tau_2 d\tau_1+$$

$$+...+\int_{t_{r-1}}^{t} A_1(\tau_1,x)...\int_{t_{r-1}}^{\tau_{\nu-2}} A_1(\tau_{\nu-1},x)\int_{t_{r-1}}^{\tau_{\nu-1}}\Big[B_1(\tau_\nu,x)\widetilde{W}_r(\tau_\nu,x)+A_2(\tau_\nu,x)\widetilde{w}_r(\tau_\nu,x)+$$

$$+\sum_{k=1}^{m} L_{1,k}(\tau_\nu,x)\widetilde{W}_{k+1}(t_k,x)\Big]d\tau_\nu d\tau_{\nu-1}...d\tau_1,$$

$$F_{\nu,r}(t,x)=\int_{t_{r-1}}^{t} F(\tau,x)d\tau+\int_{t_{r-1}}^{t} A_1(\tau_1,x)\int_{t_{r-1}}^{\tau_1} F(\tau_2,x)d\tau_2 d\tau_1+...+$$

$$+\int_{t_{r-1}}^{t} A_1(\tau_1,x)...\int_{t_{r-1}}^{\tau_{\nu-2}} A_1(\tau_{\nu-1},x)\int_{t_{r-1}}^{\tau_{\nu-1}} F(\tau_\nu,x)d\tau_\nu d\tau_{\nu-1}...d\tau_1,$$

$(t,x)\in\Omega_r$, $\quad r=\overline{1,m+1}$, $\quad \nu\in\mathbb{N}$, $\quad k=\overline{1,m}$.

From (21) we find $\widetilde{V}_r(t_r,x)=\frac{\partial\widetilde{w}_r(t_r,x)}{\partial x}$ for all $x\in[0,\omega]$, and $r=\overline{1,m+1}$. Then, substituting their into (15) and (16), and multiplying both sides (15) by $h_m=t_{m+1}-t_m$, we obtain the system of differential equations with respect to functions $\lambda_r(x)$, and $r=\overline{1,m+1}$:

$$h_m P(x)\dot{\lambda}_1(x)+h_m S(x)\sum_{k=1}^{m}\widetilde{D}_{\nu,m+1,k}(t,x)\dot{\lambda}_{k+1}(x)+$$

$$+h_m S(x)[I+D_{\nu,m+1}(t_{m+1},x)]\dot{\lambda}_{m+1}(x)=$$

$$=-h_m S(x)\Big[E_{\nu,m+1}(t_{m+1},x)\lambda_{m+1}(x)+\sum_{k=1}^{m}\widetilde{E}_{\nu,m+1,k}(t_{m+1},x)\lambda_{k+1}(x)\Big]-$$

$$-h_m S(x)G_{\nu,m+1}(t_{m+1},x,\widetilde{V}_{m+1})-h_m S(x)H_{\nu,m+1}(t_{m+1},x,\widetilde{W}_{m+1},\widetilde{w}_{m+1})-$$

$$-h_m S(x)F_{\nu,m+1}(t_{m+1},x)+h_m\varphi(x),\qquad x\in[0,\omega], \tag{22}$$

$$[I+D_{\nu,s}(t_s,x)]\dot{\lambda}_s(x)+\sum_{k=1}^{m}\widetilde{D}_{\nu,s,k}(t_s,x)\dot{\lambda}_{k+1}(x)-\dot{\lambda}_{s+1}(x)=-E_{\nu,s}(t_s,x)\lambda_s(x)-\sum_{k=1}^{m}\widetilde{E}_{\nu,s,k}(t_s,x)\lambda_{k+1}(x)-$$
$$-G_{\nu,s}(t_s,x,\widetilde{V}_s)-H_{\nu,s}(t_s,x,\widetilde{W}_s,\widetilde{w}_s)-F_{\nu,s}(t_s,x),\qquad s=\overline{1,m},\qquad x\in[0,\omega]. \tag{23}$$

We denote by $Q_\nu(x)$ and $E_\nu(x)$ the $n(m+1)\times n(m+1)$ matrices composed of the coefficients $\dot{\lambda}_r(x)$ and $\lambda_r(x)$ in (22), (23), respectively, $r=\overline{1,m+1}$.

So, we can rewrite the equations (12) and (13) in the compact form

$$Q_\nu(x)\dot{\lambda}(x)=-E_\nu(x)\lambda(x)-F_\nu(x)-G_\nu(x,\widetilde{V})-H_\nu(x,\widetilde{W},\widetilde{w}), \tag{24}$$

where $F_\nu(x)=\big(h_mS(x)F_{\nu,m+1}(t_{m+1},x)-h_m\varphi(x),F_{\nu,1}(t_1,x),...,F_{\nu,m}(t_m,x)\big)'$,

$G_\nu(x,\widetilde{V})=\big(h_mS(x)\Big[G_{\nu,m+1}(t_{m+1},x,\widetilde{V}_{m+1}),G_{\nu,1}(t_1,x,\widetilde{V}_1),...,G_{\nu,m}(t_m,x,\widetilde{V}_m)\big)'$,

$H_\nu(x,\widetilde{W},\widetilde{w})=\big(h_mS(x)H_{\nu,m+1}(t_{m+1},x,\widetilde{W}_{m+1},\widetilde{w}_{m+1}),H_{\nu,1}(t_1,x,\widetilde{W}_1,\widetilde{w}_1),...,H_{\nu,m}(t_m,x,\widetilde{W}_m,\widetilde{w}_m)\big)'$.

System (24) with conditions (17) given us Cauchy problem for ordinary differential equations with respect to $\lambda_r(x)$, $r=\overline{1,m+1}$.

If we know $\widetilde{w}_r(t,x)$ and its partial derivatives $\widetilde{V}_r(t,x)$, $\widetilde{W}_r(t,x)$, then from Cauchy problem (24), (17) we find $\dot{\lambda}_r(x)$ and $\lambda_r(x)$ for all $x\in[0,\omega]$, where $r=\overline{1,m+1}$. Conversely, if we know $\lambda_r(x)$ and its derivative $\dot{\lambda}_r(x)$, then from Goursat problem (12)–(14) we can find $\widetilde{w}_r(t,x)$ and its partial derivatives $\widetilde{V}_r(t,x)$, $\widetilde{W}_r(t,x)$ for all $(t,x)\in\Omega_r$, $r=\overline{1,m+1}$. For solve Goursat problem (12)–(14) we use equivalent system of three integral equations (18)–(20).

Since the $\widetilde{w}_r(t,x)$ and $\lambda_r(x)$ are unknown to find a solution to problem (12)–(16) we use the iterative method: 1) At fixed $\widetilde{w}_r(t,x)$ from the Cauchy problem (24), (17) we find the introducing parameters $\lambda_r(x)$ and their derivative $\dot{\lambda}_r(x)$ for all $x\in[0,\omega]$, $r=\overline{1,m+1}$; 2) At fixed $\lambda_r(x)$ from the Goursat problem (12)–(14) we find the unknown functions $\widetilde{w}_r(t,x)$ and their partial derivatives $\widetilde{V}_r(t,x)$, $\widetilde{W}_r(t,x)$ for all $(t,x)\in\Omega_r$, $r=\overline{1,m+1}$.

Let $h=\max\limits_{i=\overline{1,m+1}}(t_i-t_{i-1})$, $\alpha(x)=\max\limits_{t\in[0,T]}||A_1(t,x)||$, $\beta_k(x)=\max\limits_{t\in[0,T]}||K_{1,k}(t,x)||$, $k=\overline{1,m}$.

The following assertion given us a sufficient conditions of unique solvability to problem (12)–(16) and a convergence this iterative process.

*Theorem 1. Let for some $\nu$, $\nu\in\mathbb{N}$ the $(n(m+1)\times n(m+1))$ matrix $Q_\nu(x)$ is invertible for all $x\in[0,\omega]$ and the following conditions are valid:*

*1) $||[Q_\nu(x)]^{-1}||\le\gamma_\nu(x)$, where $\gamma_\nu(x)$ is positive and continuous on $[0,\omega]$ function;*

*2)* $q_\nu(x)=\gamma_\nu(x)\cdot\left\{e^{\alpha(x)h}-\sum\limits_{j=0}^{\nu}\frac{[\alpha(x)h]^j}{j!}+\Big[e^{\alpha(x)h}-\sum\limits_{j=0}^{\nu-1}\frac{[\alpha(x)h]^j}{j!}\Big]h\sum\limits_{k=1}^{m}\beta_k(x)\right\}\le\chi<1,$

*where $\chi$ - const.*

*Then problem with parameters (12)–(16) has unique solution.*

*Theorem 2. Let for some $\nu$, $\nu\in\mathbb{N}$ the $(n(m+1)\times n(m+1))$ matrix $Q_\nu(x)$ is invertible for all $x\in[0,\omega]$ and conditions 1)-2) of Theorem 1 are fulfilled.*

*Then nonlocal problem for system of loaded hyperbolic equations of the second order (9)–(11) has unique classical solution.*

The proofs of Theorem 1 and 2 are similar of proof Theorem 1 in [22].

Therefore, for problem (4)–(7) we have the following statement.

*Theorem 3. Let*

*i) the $n\times n$ matrices $A_i(t,x)$, $B_i(t,x)$, $K_{i,k}(t,x)$, $L_{i,k}(t,x)$, $M_{i,k}(t,x)$, $i=\overline{1,3}$, $k=\overline{1,m}$, $C(t,x)$, and $n$ vector function $f(t,x)$ are continuous on $\Omega$;*

*ii) the $n\times n$ matrices $P(x)$, $S(x)$, and $n$ vector function $\varphi(x)$ are continuous on $[0,\omega]$;*

*iii) the $n$ vector-functions $\psi_0(t)$, $\psi_1(t)$ and $\psi_2(t)$ are continuously differentiable on $[0,T]$;*

*iv) the nonlocal problem for system of loaded hyperbolic equations of the second order (9)–(11) is uniquely solvable for any $F(t,x)\in C(\Omega,\mathbb{R}^n)$, $\varphi(x)\in C([0,\omega],\mathbb{R}^n)$ and $\psi_2(t)\in C^1([0,T],\mathbb{R}^n)$.*

*Then problem with integral conditions (4)–(7) has a unique solution.*

This Theorem is proved on the basis of the above algorithm and is similar of proof Theorem 2 [23].

From equivalence of problem (1)–(3) and (4)–(7) it follows

*Theorem 4. Let*

*1) the conditions i)–iii) of Theorem 3 are fulfilled;*

*2) for some $\nu$, $\nu\in\mathbb{N}$ the $(n(m+1)\times n(m+1))$ matrix $Q_\nu(x)$ is invertible for all $x\in[0,\omega]$ and $||[Q_\nu(x)]^{-1}||\le\gamma_\nu(x)$, where $\gamma_\nu(x)$ is positive and continuous on $[0,\omega]$ function;*

*3) the following inequality holds:*

$$q_\nu(x) = \gamma_\nu(x) \cdot \left\{ e^{\alpha(x)h} - \sum_{j=0}^{\nu} \frac{[\alpha(x)h]^j}{j!} + \left[ e^{\alpha(x)h} - \sum_{j=0}^{\nu-1} \frac{[\alpha(x)h]^j}{j!} \right] h \sum_{k=1}^{m} \beta_k(x) \right\} \le \chi < 1,$$

*where $\chi$ - const.*

*Then problem (1)–(3) has a unique classical solution.*

So, the nonlocal problem for system of loaded partial differential equations of the fourth order (1)–(3) is reduced to an equivalent nonlocal problem with integral conditions for system of loaded hyperbolic equations of the second order. For solve of the nonlocal problem with integral conditions for system of loaded hyperbolic equations of the second order results of articles [22–23] are used. Algorithms of finding solutions to the nonlocal problem with integral conditions for system of loaded hyperbolic equations of the second order are constructed and their convergence is proved. The conditions of the unique solvability to the nonlocal problem for system of loaded partial differential equations of the fourth order are established.

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## Төртінші ретті дербес туындылы жүктелген дифференциалдық теңдеу үшін бейлокал есеп

Төртінші ретті дербес туындылы жүктелген дифференциалдық теңдеулер жүйесі үшін бейлокал есеп қарастырылған. Қарастырылып отырған есептің жалғыз шешімінің бар болуы мәселелері мен оны табу жолдары зерттелді. Жаңа функциялар енгізу әдісі арқылы төртінші ретті дербес туындылы жүктелген дифференциалдық теңдеулер жүйесі үшін бейлокал есеп екінші ретті жүктелген гиперболалық теңдеулер жүйесі үшін интегралдық шарттары бар бейлокал есепке келтіріледі. Нәтижесінде алынған интегралдық шарттары бар бейлокал есепті шешу үшін функционалдық параметрлер енгізу әдісі қолданылды. Жүктелген гиперболалық теңдеулер жүйесі үшін интегралдық шарттары бар бейлокал есептің жуық шешімдерін табу алгоритмдері ұсынылған және оның жинақтылығы дәлелденген. Жүктелген гиперболалық теңдеулер жүйесі үшін интегралдық шарттары бар бейлокал есептің бірмәнді шешілімділігінің шарттары бастапқы берілімдер терминінде алынған. Нәтижелер сәйкесінше бастапқы төртінші ретті дербес туындылы жүктелген дифференциалдық теңдеулер жүйесі үшін бейлокал есепке қатысты тұжырымдалған.

*Кілт сөздер:* бейлокал есеп, төртінші ретті дербес туындылы жүктелген дифференциалдық теңдеулер, интегралдық шарт, жүктелген гиперболалық теңдеулер жүйесі, алгоритм, бірмәнді шешілімділік.

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## Нелокальная задача для нагруженного дифференциального уравнения в частных производных четвертого порядка

Рассмотрена нелокальная задача для системы нагруженных дифференциальных уравнений в частных производных четвертого порядка. Исследованы вопросы существования единственного решения рассматриваемой задачи и способы его построения. Методом введения новых функций нелокальная задача для системы нагруженных дифференциальных уравнений в частных производных четвертого порядка сведена к нелокальной задаче с интегральными условиями для системы нагруженных гиперболических уравнений второго порядка. Для решения полученной нелокальной задачи с интегральными условиями применен метод введения функциональных параметров. Предложены алгоритмы нахождения приближенного решения нелокальной задачи с интегральными условиями для системы нагруженных гиперболических уравнений второго порядка и доказана их сходимость. Получены условия однозначной разрешимости нелокальной задачи с интегральными условиями для системы нагруженных гиперболических уравнений в терминах исходных данных. Результаты сформулированы относительно исходной нелокальной задачи для системы нагруженных дифференциальных уравнений в частных производных четвертого порядка.

*Ключевые слова:* нелокальная задача, нагруженные дифференциальные уравнения в частных производных четвертого порядка, интегральное условие, система нагруженных гиперболических уравнений, алгоритм, однозначная разрешимость.

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# Order of the trigonometric widths of the Nikol'skii-Besov classes with mixed metric in the metric of anisotropic Lorentz spaces

In this paper we estimate the order of the triginometric width of the Nikol'skii–Besov classes $B_{\mathbf{p}}^{\alpha\tau}(\mathbb{T}^n)$ with mixed metric in the anisotropic Lorentz space $L_{\mathbf{q}\theta}(\mathbb{T}^n)$ when $\mathbf{1}<\mathbf{p}=(p_1,\ldots,p_n)<\mathbf{2}<\mathbf{q}=(q_1,\ldots,q_n)$. The concept of a trigonometric width in the one-dimensional case was first introduce by R.S. Ismagilov and he established his estimates for certain classes in the space of continuous functions. For a function of several variables exact orders of trigonometric width of Sobolev class $W_p^{\mathbf{r}}$, Nikol'skii class $H_p^{\mathbf{r}}$ in the space $L_q$ are established by E.S. Belinsky, V.E. Majorov, Yu. Makovoz, G.G. Magaril-Ilyaev, V.N. Temlyakov. This problem for the Besov class $B_{pq}^{\mathbf{r}}$ was investigated by A.S. Romanyuk, D.B. Bazarkhanov. The trigonometric width for the anisotropic Nikol'skii-Besov classes $B_{\mathbf{pr}}^{\alpha\tau}(\mathbb{T}^n)$ in the metric of the anisotropic Lorentz spaces $L_{\mathbf{q}\theta}(\mathbb{T}^n)$ was found by K.A. Bekmaganbetov and Ye. Toleugazy.

*Keywords:* trigonometric widths, anisotropic Lorentz space, Nikol'skii–Besov class with mixed metric.

*Introduction*

Let $V\subset L_1(\mathbb{T}^n)$ be the normed space and $F\subset V$ be some functional class. The trigonometric width of the class $F$ in the space $V$ is defined as follows (see [1])

$$d_M^T(F,V)=\inf_{\Omega_M}\sup_{f\in F}\inf_{t(\Omega_M;\mathbf{x})}\|f(\cdot)-t(\Omega_M;\cdot)\|_V\,,$$

where $t(\Omega_M;\mathbf{x})=\sum_{j=1}^{M}c_j\mathrm{e}^{i(\mathbf{k}_j,\mathbf{x})}$, $\Omega_M=\{\mathbf{k}_1,\ldots,\mathbf{k}_M\}$ is the set of vectors $\mathbf{k}_j=(k_1^j,\ldots,k_n^j)$ from the integer lattice $\mathbb{Z}^n$ and $c_j$ are some numbers $(j=1,\ldots,M)$.

The concept of a trigonometric width in the one-dimensional case was first introduced by R.S. Ismagilov [1] and he established its estimates for certain classes in the space of continuous functions. For a function of several variables exact orders of trigonometric widths of Sobolev class $W_p^{\mathbf{r}}$, Nikol'skii class $H_p^{\mathbf{r}}$ in the space $L_q$ are established by E.S. Belinsky [2], V.E. Majorov [3], Yu. Makovoz [4], G.G. Magaril-Ilyaev [5], V.N. Temlyakov [6]. This problem for the Besov class $B_{pq}^{\mathbf{r}}$ was investigated by A.S. Romanyuk [7], D.B. Bazarkhanov [8]. The trigonometric width for the anisotropic Nikol'skii-Besov classes $B_{\mathbf{pr}}^{\alpha\tau}(\mathbb{T}^n)$ in the metric of the anisotropic Lorentz spaces $L_{\mathbf{q}\theta}(\mathbb{T}^n)$ was found by K.A. Bekmaganbetov and Ye. Toleugazy [9].

We study the problem of estimating the order of the trigonimetric width of the Nikol'skii-Besov classes $B_{\mathbf{p}}^{\alpha\tau}(\mathbb{T}^n)$ with a mixed metric in the metric of anisotropic Lorentz spaces $L_{\mathbf{q}\theta}(\mathbb{T}^n)$.

*Preliniminaries and auxiliary results*

Let $f(\mathbf{x})=f(x_1,\ldots,x_n)$ be a measurable function defined by $\mathbb{T}^n$. Let multiindexes $\mathbf{1}\le\mathbf{p}=(p_1,\ldots,p_n)<<\infty$. A Lebesgue space $L_{\mathbf{p}}(\mathbb{T}^n)$ with mixed metric is the set of functions for which the following quantity is finite

$$\|f\|_{L_{\mathbf{p}}(\mathbb{T}^n)}=\left(\int_0^{2\pi}\left(\ldots\left(\int_0^{2\pi}|f(x_1,\ldots,x_n)|^{p_1}\mathrm{d}x_1\right)^{p_2/p_1}\ldots\right)^{p_n/p_{n-1}}\mathrm{d}x_n\right)^{1/p_n}.$$

Here, the expression $\left(\int_0^{2\pi}|f(t)|^p\mathrm{d}t\right)^{1/p}$ for $p{=}\infty$ is understood as $\sup_{0\le t<2\pi}|f(t)|$.

For the function $f \in L_{\mathbf{p}}(\mathbb{T}^n)$ we denote

$$\Delta_{\mathbf{s}}(f, \mathbf{x}) = \sum_{\mathbf{k}\in\rho(\mathbf{s})} a_{\mathbf{k}}(f) e^{i(\mathbf{k},\mathbf{x})},$$

where $\{a_{\mathbf{k}}(f)\}_{\mathbf{k}\in\mathbb{Z}^n}$ are Fourier coefficients of the function $f$ with respect to the multiple trigonometric system $\rho(\mathbf{s}) = \{\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{Z}^n : 2^{s_i-1} \leq |k_i| < 2^{s_i}, i = 1, \ldots, n\}$, $(\mathbf{k}, \mathbf{x}) = \sum_{j=1}^n k_j x_j$ – inner product.

Let $\mathbf{0} < \alpha = (\alpha_1, \ldots, \alpha_n) < \infty$, $\mathbf{0} < \tau = (\tau_1, \ldots, \tau_n) \leq \infty$. The class of Nikol'skii-Besov $B_{\mathbf{p}}^{\alpha\tau}(\mathbb{T}^n)$ with a mixed metric is the set of functions $f$ from $L_{\mathbf{p}}(\mathbb{T}^n)$ for which the following inequality holds

$$\|f\|_{B_{\mathbf{p}}^{\alpha\tau}(\mathbb{T}^n)} = \left\| \left\{ \mathbf{2}^{(\alpha,\ \mathbf{s})} \|\triangle_{\mathbf{s}}(f, \cdot)\|_{L_{\mathbf{p}}(\mathbb{T}^n)} \right\}_{\mathbf{s}\in\mathbb{Z}_+^n} \right\|_{l_\tau} \leq 1,$$

where $\|\cdot\|_{l_\tau}$ is the norm of discrete Lebesgue space $l_\tau$ with a mixed metric.

Let $f(\mathbf{x}) = f(x_1, \ldots, x_n)$ be a measurable function defined on $\mathbb{T}^n$. We denote by $f^*(\mathbf{t}) = f^{*_1,\ldots,*_n}(t_1, \ldots, t_n)$ the function obtained by applying to the first nonicreasing permutation, successively with respect to the variables $x_1, \ldots, x_n$ for fixed other variables.

Let multiindexes $\mathbf{q} = (q_1, \ldots, q_n)$, $\theta = (\theta_1, \ldots, \theta_n)$ satisfy the conditions: if $0 < q_j < \infty$, then $0 < \theta_j \leq \infty$, if $q_j = \infty$, then $\theta_j = \infty$ for every $j = 1, \ldots, n$. An anisotropic Lorentz space $L_{\mathbf{q}\theta}(\mathbb{T}^n)$ is the set of functions for which the following quantity is finite

$$\|f\|_{L_{\mathbf{q}\theta}(\mathbb{T}^n)} = \left( \int_0^{2\pi} \ldots \left( \int_0^{2\pi} \left( t_1^{1/q_1} \ldots t_n^{1/q_n} f^{*_1,\ldots,*_n}(t_1, \ldots, t_n) \right)^{\theta_1} \frac{dt_1}{t_1} \right)^{\theta_2/\theta_1} \ldots \frac{dt_n}{t_n} \right)^{1/\theta_n}.$$

Let $\Omega_M$ be a set containing at most $M$ vectors $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{Z}^n$.

*Lemma 1* [10]. Let $\mathbf{2} \leq \mathbf{q} < \infty$. Then for any trigonometric polynomial

$$P(\Omega_M, \mathbf{x}) = \sum_{j=1}^M e^{i(\mathbf{k}^j,\mathbf{x})}$$

and for any number $N \leq M$ there exists a trigonometric polynomial $P(\Omega_N, \mathbf{x})$ containing at most $N$ harmonics and such that

$$\|P(\Omega_M, \cdot) - P(\Omega_N, \cdot)\|_{L_{\mathbf{q}}(\mathbb{T}^n)} \leq CMN^{-1/2},$$

moreover $\Omega_N \subset \Omega_M$ and all coefficients $P(\Omega_N, \mathbf{x})$ are the same and do not exceed $MN^{-1}$.

*Corollary 1* [11]. Let $\mathbf{2} < \mathbf{q} = (q_1, \ldots, q_n) \leq \infty$, $\mathbf{0} < \theta = (\theta_1, \ldots, \theta_n) \leq \infty$. Then for any trigonometric polynomial

$$P(\Omega_M, \mathbf{x}) = \sum_{j=1}^M e^{i(\mathbf{k}^j,\mathbf{x})}$$

and for any number $N \leq M$ there exists a trigonometric polynomial $P(\Omega_N, \mathbf{x})$ containing at most $N$ harmonics and such that

$$\|P(\Omega_M, \cdot) - P(\Omega_N, \cdot)\|_{L_{\mathbf{q}\theta}(\mathbb{T}^n)} \leq CMN^{-1/2},$$

moreover $\Omega_N \subset \Omega_M$ and all coefficients $P(\Omega_N, \mathbf{x})$ are the same and do not exceed $MN^{-1}$.

For any $\mathbf{s} \in \mathbb{Z}_+^n$ we consider a linear operator

$$\left(T_{N_{\mathbf{s}}} f\right)(\mathbf{x}) = f(\mathbf{x}) * \left( \sum_{\mathbf{k}\in\rho(\mathbf{s})} e^{i(\mathbf{k},\mathbf{x})} - t(\Omega_{N_{\mathbf{s}}}, \mathbf{x}) \right),$$

where $t(\Omega_{N_{\mathbf{s}}}, \mathbf{x})$ is a trigonometric polynomial from Corollary 1, which is approaching the «block» $t_{\mathbf{s}}(\mathbf{x}) = \sum\limits_{\mathbf{k}\in\rho(\mathbf{s})} e^{i(\mathbf{k},\mathbf{x})}$.

*Lemma 2.* Let $\mathbf{1} < \mathbf{p} < \mathbf{2}$, the multiindex $\mathbf{q} = (q_1, \ldots, q_n)$ be such that $2 < q_j < p'$ for all $j = 1, \ldots, n$ and $\mathbf{0} < \theta = (\theta_1, \ldots, \theta_n) \leq \infty$. Then the norm operator $T_{N_{\mathbf{s}}}$ acting from $L_{\mathbf{p}}(\mathbb{T}^n)$ to $L_{\mathbf{q}\theta}(\mathbb{T}^n)$ satisfies the following inequality

$$\|T_{N_{\mathbf{s}}}\|_{L_{\mathbf{p}}(\mathbb{T}^n)\to L_{\mathbf{q}\theta}(\mathbb{T}^n)} \leq C_1 2^{(\mathbf{1},\mathbf{s})} N_{\mathbf{s}}^{-(\mathbf{1/2}+\mathbf{1/p'})}.$$

*Proof.* Taking into account that the coefficients of the polynomial $t(\Omega_{N_{\mathbf{s}}}, \mathbf{x})$ are the same and do not exceed $2^{(\mathbf{1},\mathbf{s})}N_{\mathbf{s}}^{-1}$ by Parseval's equality we have

$$\|T_{N_{\mathbf{s}}}\|_{L_{\mathbf{2}}(\mathbb{T}^n)\to L_{\mathbf{2}}(\mathbb{T}^n)} \le C_1 2^{(\mathbf{1},\mathbf{s})}N_{\mathbf{s}}^{-1}. \tag{1}$$

Further, using the generalized Minkowski's inequalities and Corollary 1 we can write

$$\|T_{N_{\mathbf{s}}}f\|_{L_{\mathbf{q}^*\theta^*}(\mathbb{T}^n)} \le \|f\|_{L_1(\mathbb{T}^n)}\left\|\sum_{\mathbf{k}\in\rho(\mathbf{s})} e^{i(\mathbf{k},\cdot)} - t(\Omega_{N_{\mathbf{s}}},\cdot)\right\|_{L_{\mathbf{q}^*\theta^*}(\mathbb{T}^n)} \le C_2 2^{(\mathbf{1},\mathbf{s})}N_{\mathbf{s}}^{-1/2}\|f\|_{L_{\mathbf{1}}(\mathbb{T}^n)}.$$

Therefore, by definition, $\|T_{N_{\mathbf{s}}}\|_{L_{\mathbf{1}}(\mathbb{T}^n)\to L_{\mathbf{q}^*\theta^*}(\mathbb{T}^n)}$ we find

$$\|T_{N_{\mathbf{s}}}\|_{L_{\mathbf{1}}(\mathbb{T}^n)\to L_{\mathbf{q}^*\theta^*}(\mathbb{T}^n)} \le C_2 2^{(\mathbf{1},\mathbf{s})}N_{\mathbf{s}}^{-1/2}. \tag{2}$$

Further, using the Riesz-Thorin interpolation theorem for Lebegue spaces and anisotropic Lorentz spaces, we obtain

$$\|T_{N_{\mathbf{s}}}\|_{L_{\mathbf{1}}(\mathbb{T}^n)\to L_{\mathbf{q}^*\theta^*}(\mathbb{T}^n)} \le \|T_{N_{\mathbf{s}}}\|^{1-\lambda}_{L_{\mathbf{2}}(\mathbb{T}^n)\to L_{\mathbf{2}}(\mathbb{T}^n)}\|T_{N_{\mathbf{s}}}\|^{\lambda}_{L_{\mathbf{1}}(\mathbb{T}^n)\to L_{\mathbf{q}^*\theta^*}(\mathbb{T}^n)}, \tag{3}$$

where $\mathbf{0} < \lambda < \mathbf{1}$ and $\mathbf{1}/\mathbf{p} = (\mathbf{1}-\lambda)/\mathbf{2} + \lambda/\mathbf{1}$, $\mathbf{1}/\mathbf{q}$=$(\mathbf{1}-\lambda)/\mathbf{2}$+$\lambda/\mathbf{q}^*$ and $\mathbf{1}/\theta = (\mathbf{1}-\lambda)/\mathbf{2} + \lambda/\theta^*$.

By substuting (1) and (2) to (3) and performing elementary transformations, we reccieve at the required estimate with the additional condition $\mathbf{0} < \theta = (\theta_1,\ldots,\theta_n) < \mathbf{p}' = (p',\ldots,p')$. For the remaining values of the parameters $\theta = (\theta_1,\ldots,\theta_n)$ the validy of the assertion follows from the embedding $L_{\mathbf{q}\theta_{\mathbf{1}}}(\mathbb{T}^n) \hookrightarrow L_{\mathbf{q}\theta_{\mathbf{2}}}(\mathbb{T}^n)$ for $\mathbf{0} < \theta_1 = (\theta_1^1,\ldots,\theta_n^1) \le \theta_2 = (\theta_1^2,\ldots,\theta_n^2) \le \infty$.

Let us formulate a special case of the embedding theorem from E.D. Nursultanov's paper ([12]) as a Lemma.

*Lemma 3* [12]. Let $\mathbf{1} \le \mathbf{p} = (p_1,\ldots,p_n) < \mathbf{q} = (q_1,\ldots,q_n) < \infty$, $\mathbf{0} < \tau = (\tau_1,\ldots,\tau_n) \le \infty$ and $\alpha = \mathbf{1}/\mathbf{p} - \mathbf{1}/\mathbf{q}$, then

$$B^{\alpha\tau}_{\mathbf{p}}(\mathbb{T}^n) \hookrightarrow L_{\mathbf{q}\tau}(\mathbb{T}^n).$$

Furthermore we need the following sets

$$Y^n(N,\gamma) = \left\{\mathbf{s} = (s_1,\ldots,s_n) \in \mathbb{Z}^n_+ : \sum_{j=1}^n \gamma_j s_j \ge N\right\},$$

$$\aleph^n(N,\gamma) = \left\{\mathbf{s} = (s_1,\ldots,s_n) \in \mathbb{Z}^n_+ : \sum_{j=1}^n \gamma_j s_j = N\right\}.$$

*Lemma 4 [13].* Let $n \in \mathbb{N}$, $n \ge 2$, $\mathbf{0} < \gamma' = (\gamma_1',\ldots,\gamma_n') \le \gamma = (\gamma_1,\ldots,\gamma_n) < \infty$, $\delta > 0$ and $\mathbf{0} < \varepsilon = (\varepsilon_1,\ldots,\varepsilon_n) \le \infty$. Then

$$\left\|\left\{\mathbf{2}^{-\delta(\gamma,\ \mathbf{s})}\right\}_{\mathbf{s}\in Y^n(N,\ \gamma')}\right\|_{l_\varepsilon(\mathbb{Z}^n_+)} \le C2^{-\delta\eta N}N^{\sum_{j\in A\setminus\{j_1\}} 1/\varepsilon_j},$$

where $\eta = \min\{\gamma_j/\gamma_j' : j = 1,\ldots,n\}$, $A = \{j : \gamma_j/\gamma_j' = \eta, j = 1,\ldots,n\}$, $j_1 = \min\{j : j \in A\}$.

*Lemma 5* [13]. Let $n \in \mathbb{N}$, $n \ge 2$, $\mathbf{0} < \gamma = (\gamma_1,\ldots,\gamma_n) < \infty$, $\delta \in \mathbb{R}$ and $\mathbf{0} < \varepsilon = (\varepsilon_1,\ldots,\varepsilon_n) \le \infty$. Then

$$\left\|\left\{\mathbf{2}^{-\delta(\gamma,\ \mathbf{s})}\right\}_{\mathbf{s}\in \aleph^n(N,\ \gamma')}\right\|_{l_\varepsilon(\mathbb{Z}^n_+)} \asymp 2^{-\delta N}N^{\sum_{j=2}^n 1/\varepsilon_j}.$$

## *Main result*

The main result of this paper includes:

*Theorem 1.* Let $\mathbf{1} < \mathbf{p} = (p_1,\ldots,p_n) < \mathbf{2} < \mathbf{q} = (q_1,\ldots,q_n) < \mathbf{p_0'} = (p_0',\ldots,p_0')$, $p_0 = \max\{p_j : j = 1,\ldots,n\}$, $\mathbf{1} \le \tau = (\tau_1,\ldots,\tau_n), \theta = (\theta_1,\ldots,\theta_n)$ and $\alpha = (\alpha_1,\ldots,\alpha_n)$ be such that $\alpha_j > 1 + 1/p_j - 1/p_0$ for all $j = 1,\ldots,n$. Let $\zeta = \min\{\alpha_j - 1/p_j + 1/q_j : j = 1,\ldots,n\}$, $D = \{j = 1,\ldots,n : \alpha_j - 1/p_j + 1/q_j = \zeta\}$, $j_1 = \min\{j : j \in D\}$, $q_j = q_{j_1}$ for all $j \in D$ and $q_j \ge q_{j_1}$ for all $j \notin D$.

Then the following relation holds

$$d_M^T(B_{\mathbf{p}}^{\alpha\tau}(\mathbb{T}^n),L_{\mathbf{q}\theta}(\mathbb{T}^n)) \asymp M^{-(\alpha_{j_1}-1/p_{j_1}+1/2)}(\log M)^{(|D|-1)(\alpha_{j_1}-1/p_{j_1}+1/2)+\sum_{j\in D\setminus\{j_1\}}(1/2-1/\tau_j)_+}, \quad (4)$$

where $|D|$ is amount of elements of the set $D$, $a_+ = \max\{a;0\}$.

*Proof.* Let $f \in B_{\mathbf{p}}^{\alpha\tau}(\mathbb{T}^n)$. For any natural number $M$ there exists the natural number $m$ such that $M \asymp 2^m m^{(|D|-1)}$. We will seek an approximating polynomial $P(\Omega_M, \mathbf{x})$ in the following form

$$P(\Omega_M;\mathbf{x}) = \sum_{(\gamma',\mathbf{s})} \triangle_{\mathbf{s}}(f,\mathbf{x}) + \sum_{m\le(\gamma',\mathbf{s})<\beta m} t(\Omega_{N_{\mathbf{s}}};\mathbf{x}) * \triangle_{\mathbf{s}}(f,\mathbf{x}), \quad (5)$$

where

$$\beta = \left(\alpha_{j_1} - 1/p_{j_1} + 1/2 - \frac{\log m}{m} \sum_{j\in D\setminus\{j_1\}} \Big((1/2-1/\tau_j)_+ - (1/\theta_j - 1/\tau_j)_+\Big)\right) / (\alpha_{j_1} - 1/p_{j_1} + 1/q_{j_1}),$$

$\gamma_j = (\alpha_j - 1/p_j + 1/q_j)/(\alpha_{j_1} - 1/p_{j_1} + 1/q_{j_1})$, $j = 1,\ldots,n$, $\gamma'_j = \gamma_j$ for $j \in D$ and $1 < \gamma'_j < \gamma_j$ for $j \notin D$. The polynomials $t(\Omega_{N_{\mathbf{s}}}, \mathbf{x})$ are chosen for every "block" $t_{\mathbf{s}}(\mathbf{x}) = \sum_{\mathbf{k}\in\rho(\mathbf{s})} e^{i(\mathbf{k},\mathbf{x})}$ according to Corollary 1 and numbers $N_{\mathbf{s}} = \left[2^{(\alpha_{j_1}-1/p_{j_1}+1/p_0)m} 2^{-(\alpha-\mathbf{1/p+1/p_0-1},\mathbf{s})}\right]$.

Note that according to Lemma 4

$$\sum_{m\le(\gamma',\mathbf{s})<\beta m} N_{\mathbf{s}} = 2^{\left(\alpha_{j_1}-1/p_{j_1}+1/p_0\right)m} \sum_{m\le(\gamma',\mathbf{s})<\beta m} 2^{-(\alpha-\mathbf{1/p+1/p_0-1},\mathbf{s})} \le$$

$$\le 2^{\left(\alpha_{j_1}-1/p_{j_1}+1/p_0\right)m} \left\| \left\{2^{-(\alpha-\mathbf{1/p+1/p_0-1},\mathbf{s})}\right\}_{\mathbf{s}\in Y^n\left(m,\gamma'\right)} \right\|_{l_1} \le$$

$$\le 2^{\left(\alpha_{j_1}-1/p_{j_1}+1/p_0\right)m} 2^{-\left(\alpha_{j_1}-1/p_{j_1}+1/p_0\right)m} m^{(|D|-1)} = 2^m m^{(|D|-1)} \asymp M,$$

so that $(\alpha_j - 1/p_j + 1/p_0 - 1)/(\alpha_{j_1} - 1/p_{j_1} + 1/p_0 - 1) > \gamma'_j$ at $j \notin D$.

Moreover according to equality (5) and Minkowski's inequality we have

$$\|f(\cdot) - P(\Omega_M;\cdot)\|_{L_{\mathbf{q}\theta}(\mathbb{T}^n)} \le$$

$$\le C_1 \left( \left\| \sum_{m\le(\gamma',\mathbf{s})<\beta m} (\triangle_{\mathbf{s}}(f,\cdot) - \triangle_{\mathbf{s}}(f,\cdot) * t(\Omega_{N_{\mathbf{s}}};\cdot)) \right\|_{L_{\mathbf{q}\theta}(\mathbb{T}^n)} + \left\| \sum_{(\gamma',\mathbf{s})\ge\beta m} \triangle_{\mathbf{s}}(f,\cdot) \right\|_{L_{\mathbf{q}\theta}(\mathbb{T}^n)} \right) =$$

$$= C_1 (I_1(f) + I_2(f)). \quad (6)$$

Firstly we estimate $I_2(f)$. By Lemma 3 we have

$$I_2(f) \le C_2 \left\| \left\{ 2^{(\mathbf{1/p-1/q},\mathbf{s})} \|\triangle_{\mathbf{s}}(f,\cdot)\|_{L_{\mathbf{p}}(\mathbb{T}^n)} \right\}_{\mathbf{s}\in Y^n\left(\beta m,\gamma'\right)} \right\|_{l_\theta} \le I_3(f). \quad (7)$$

According to Hölder's inequality with parameters $\mathbf{1}/\theta = \mathbf{1}/\tau + \mathbf{1}/\varepsilon$, where $\mathbf{1}/\varepsilon = (\mathbf{1}/\theta - \mathbf{1}/\tau)_+$ and Lemma 4, taking into account that $\gamma' \le \gamma$ we find

$$I_3(f) = \left\| \left\{ 2^{(\alpha,\mathbf{s})} \|\triangle_{\mathbf{s}}(f,\cdot)\|_{L_{\mathbf{p}}(\mathbb{T}^n)} \cdot 2^{-\left(\alpha_{j_1}-1/p_{j_1}+1/q_{j_1}\right)(\gamma,\mathbf{s})} \right\}_{\mathbf{s}\in Y^n\left(\beta m,\gamma'\right)} \right\|_{l_\theta} \le$$

$$\le \left\| \left\{ 2^{(\alpha,\mathbf{s})} \|\triangle_{\mathbf{s}}(f,\cdot)\|_{L_{\mathbf{p}}(\mathbb{T}^n)} \right\}_{\mathbf{s}\in Y^n\left(\beta m,\gamma'\right)} \right\|_{l_\tau} \times$$

$$\times \left\| \left\{ 2^{-\left(\alpha_{j_1}-1/p_{j_1}+1/q_{j_1}\right)(\gamma,\mathbf{s})} \right\}_{\mathbf{s}\in Y^n\left(\beta m,\gamma'\right)} \right\|_{l_\varepsilon} \le$$

$$\leq C_3\|f\|_{B_{\mathbf{p}}^{\alpha\tau}(\mathbb{T}^n)}\cdot 2^{-\left(\alpha_{j_1}-1/p_{j_1}+1/q_{j_1}\right)\beta m}m^{\sum_{j\in D\setminus\{j_1\}}1/\varepsilon_j}\leq$$

$$\leq C_3 2^{-\left(\alpha_{j_1}-1/p_{j_1}+1/q_{j_1}\right)\beta m}m^{\sum_{j\in D\setminus\{j_1\}}(1/\theta_j-1/\tau_j)_+}. \tag{8}$$

Inserting (8) into (7) we have

$$I_2\leq C_4 2^{-\left(\alpha_{j_1}-1/p_{j_1}+1/q_{j_1}\right)\beta m}m^{\sum_{j\in D\setminus\{j_1\}}(1/\theta_j-1/\tau_j)_+}.$$

Next by using $\beta$ we obtain

$$2^{-\left(\alpha_{j_1}-1/p_{j_1}+1/q_{j_1}\right)\beta m}=2^{-\left(\alpha_{j_1}-1/p_{j_1}+1/2\right)\beta m}m^{\sum_{j\in D\setminus\{j_1\}}(1/2-1/\tau_j)_+},$$

and consequently

$$I_2(f)\leq C_4 2^{-\left(\alpha_{j_1}-1/p_{j_1}+1/2\right)\beta m}m^{\sum_{j\in D\setminus\{j_1\}}(1/2-1/\tau_j)_+}.$$

Taking into account that $M\asymp 2^m m^{(|D|-1)}$ we have

$$I_2(f)\leq C_5 M^{-\left(\alpha_{j_1}-1/p_{j_1}+1/2\right)}(\log M)^{(|D|-1)\left(\alpha_{j_1}-1/p_{j_1}+1/2\right)+\sum_{j\in D\setminus\{j_1\}}(1/2-1/\tau_j)_+}. \tag{9}$$

Now, let us estimate the value $I_1(f)$. By using the Littlewood-Paley theorem (see [14]), we obtain

$$I_1(f)=\left\|\sum_{m\leq(\gamma',\mathbf{s})<\beta m}\left(\triangle_{\mathbf{s}}(f,\cdot)-\triangle_{\mathbf{s}}(f,\cdot)\ast t(\Omega_{N_{\mathbf{s}}};\cdot)\right)\right\|_{L_{\mathbf{q}\theta}(\mathbb{T}^n)}\leq$$

$$\leq C_6\left(\left\|\sum_{m\leq(\gamma',\mathbf{s})<\beta m}\left(\triangle_{\mathbf{s}}(f,\cdot)-\triangle_{\mathbf{s}}(f,\cdot)\ast t(\Omega_{N_{\mathbf{s}}};\cdot)\right)\right\|^2_{L_{\mathbf{q}\theta}(\mathbb{T}^n)}\right)^{1/2}\leq$$

$$\leq C_6\left\|\left\{\left\|\triangle_{\mathbf{s}}(f,\cdot)\ast\left(\sum_{\mathbf{k}\in\rho(\mathbf{s})}e^{i(\mathbf{k},\cdot)}-t(\Omega_{N_{\mathbf{s}}},\cdot)\right)\right\|_{L_{\mathbf{q}\theta}(\mathbb{T}^n)}\right\}_{\mathbf{s}\in\aleph^n\left(m,\beta m,\gamma'\right)}\right\|_{l_{\mathbf{2}}}=$$

$$=C_6\left\|\left\{\|T_{\mathbf{s}}\triangle_{\mathbf{s}}(f,\cdot)\|_{L_{\mathbf{q}\theta}(\mathbb{T}^n)}\right\}_{\mathbf{s}\in\aleph^n\left(m,\beta m,\gamma'\right)}\right\|_{l_{\mathbf{2}}}, \tag{10}$$

where $\aleph^n(m,\beta m,\gamma')=\{\mathbf{s}\in\mathbb{Z}_+^n: m\leq(\gamma',\mathbf{s})<\beta m\}$.

By using Lemma 2 and inequality of different metric for trigonometric polynomials in the Lebesgue spaces with mixed metric (see [14]) for $1<p_j<p_0$ $(j=1,\ldots,n)$, from (10) we have

$$I_1(f)\leq C_7\left\|\left\{2^{(\mathbf{1},\mathbf{s})}N_{\mathbf{s}}^{-\left(1/2+1/p_0'\right)}\|\triangle_{\mathbf{s}}(f,\cdot)\|_{L_{\mathbf{p_0}}(\mathbb{T}^n)}\right\}_{\mathbf{s}\in\aleph^n\left(m,\beta m,\gamma'\right)}\right\|_{l_{\mathbf{2}}}\leq$$

$$\leq C_8\left\|\left\{2^{(\mathbf{1},\mathbf{s})}N_{\mathbf{s}}^{-\left(1/2+1/p_0'\right)}2^{(\mathbf{1/p+1/p_0},\mathbf{s})}\|\triangle_{\mathbf{s}}(f,\cdot)\|_{L_{\mathbf{p}}(\mathbb{T}^n)}\right\}_{\mathbf{s}\in\aleph^n\left(m,\beta m,\gamma'\right)}\right\|_{l_{\mathbf{2}}}=$$

$$=C_8\left\|\left\{N_{\mathbf{s}}^{-\left(1/2+1/p_0'\right)}2^{-(\alpha-\mathbf{1/p+1/p_0-1},\mathbf{s})}\cdot 2^{(\alpha,\mathbf{s})}\|\triangle_{\mathbf{s}}(f,\cdot)\|_{L_{\mathbf{p}}(\mathbb{T}^n)}\right\}_{\mathbf{s}\in\aleph^n\left(m,\beta m,\gamma'\right)}\right\|_{l_{\mathbf{2}}}. \tag{11}$$

According to Hölder's inequality with parameters $\mathbf{1/2}=\mathbf{1}/\tau+\mathbf{1}/\varepsilon$, where $\mathbf{1}/\varepsilon=(\mathbf{1/2}-\mathbf{1}/\tau)_+$ and by (11) we find

$$I_1(f)\leq C_8\left\|\left\{2^{(\alpha,\mathbf{s})}\|\triangle_{\mathbf{s}}(f,\cdot)\|_{L_{\mathbf{p}}(\mathbb{T}^n)}\right\}_{\mathbf{s}\in\aleph^n\left(m,\beta m,\gamma'\right)}\right\|_{l_\tau}\times$$

$$\times\left\|\left\{N_{\mathbf{s}}^{-\left(1/2+1/p_0'\right)}2^{-(\alpha-\mathbf{1/p+1/p_0-1},\mathbf{s})}\right\}_{\mathbf{s}\in\aleph^n\left(m,\beta m,\gamma'\right)}\right\|_{l_\varepsilon}\leq$$

$$\leq C_8\|f\|_{B^{\alpha\tau}_{\mathbf{p}}(\mathbb{T}^n)}\left\|\left\{N_{\mathbf{s}}^{-\left(1/2+1/p_0'\right)}2^{-(\alpha-\mathbf{1/p+1/p_0-1,s})}\right\}_{\mathbf{s}\in\aleph^n\left(m,\beta m,\gamma'\right)}\right\|_{l_\varepsilon}\leq$$

$$\leq C_8\left\|\left\{N_{\mathbf{s}}^{-\left(1/2+1/p_0'\right)}2^{-(\alpha-\mathbf{1/p+1/p_0-1,s})}\right\}_{\mathbf{s}\in\aleph^n\left(m,\beta m,\gamma'\right)}\right\|_{l_\varepsilon} \tag{12}$$

for any function $f\in B^{\alpha\tau}_{\mathbf{p}}(\mathbb{T}^n)$.

By continuing (12), according to the Lemma 4 we have

$$I_1(f)\leq C_8 2^{-\left(1/2+1/p_0'\right)\left(\alpha_{j_1}-1/p_{j_1}+1/p_0\right)m}\times$$

$$\times\left\|\left\{2^{\left(1/2+1/p_0'\right)(\alpha-\mathbf{1/p+1/p_0-1,s})}\cdot 2^{-(\alpha-\mathbf{1/p+1/p_0-1,s})}\right\}_{\mathbf{s}\in\aleph^n\left(m,\beta m,\gamma'\right)}\right\|_{l_\varepsilon}=$$

$$=C_8 2^{-\left(1/2+1/p_0'\right)\left(\alpha_{j_1}-1/p_{j_1}+1/p_0\right)m}\times$$

$$\times\left\|\left\{2^{-\left(1/2+1/p_0'\right)(\alpha-\mathbf{1/p+1/p_0-1,s})}\right\}_{\mathbf{s}\in\aleph^n\left(m,\beta m,\gamma'\right)}\right\|_{l_\varepsilon}\leq$$

$$\leq C_8 2^{-\left(1/2+1/p_0'\right)\left(\alpha_{j_1}-1/p_{j_1}+1/p_0\right)m}\left\|\left\{2^{-\left(1/2+1/p_0'\right)(\alpha-\mathbf{1/p+1/p_0-1,s})}\right\}_{\mathbf{s}\in Y^n\left(m,\gamma'\right)}\right\|_{l_\varepsilon}\leq$$

$$\leq C_9 2^{-\left(1/2+1/p_0'\right)\left(\alpha_{j_1}-1/p_{j_1}+1/p_0\right)m}2^{-\left(1/2+1/p_0'\right)\left(\alpha_{j_1}-\frac{1}{p_{j_1}}+\frac{1}{p_0}-1\right)m}m^{\sum_{j\in D\setminus\{j_1\}}1/\varepsilon_j}=$$

$$=C_9 2^{-\left(\alpha_{j_1}-1/p_{j_1}+1/2\right)m}m^{\sum_{j\in D\setminus\{j_1\}}(1/2-1/\tau_j)_+},$$

as $(\alpha_j-1/p_j+1/p_0-1)\,/\,(\alpha_{j_1}-1/p_{j_1}+1/p_0-1)>\gamma_j'$ at $j\notin D$.

Taking into account that $M\asymp 2^m m^{(|D|-1)}$ we find

$$I_2(f)\leq C_{10}M^{-\left(\alpha_{j_1}-1/p_{j_1}+1/2\right)}(\log M)^{(|D|-1)\left(\alpha_{j_1}-1/p_{j_1}+1/2\right)+\sum_{j\in D\setminus\{j_1\}}(1/2-1/\tau_j)_+}. \tag{13}$$

Inserting (9) and (13) into (6) we obtain the inequality, which gives the upper estimate in (4).

For the proof of the lower estimate we consider the following value

$$e_M(F)_V=\sup_{f\in F}\inf_{\{b_j,\mathbf{k}_j\}_{j=1}^M}\left\|f-\sum_{j=1}^M b_j e^{i(\mathbf{k}_j,\mathbf{x})}\right\|_V,$$

which is called the best $M$-term appoximation of the class $F$ in metric space $V$.

Moreover, by the definition, the following inequality holds

$$e_M(F)_V\leq d_M^T(F,V).$$

By using the condition $2<q_j$ $(j=1,\ldots,n)$ we have

$$e_M(f)_{L_{\mathbf{2}}(\mathbb{T}^n)}\leq C_{11}e_M(f)_{L_{\mathbf{q}\theta}(\mathbb{T}^n)}.$$

For the proof of the lower estimate we will use double relation, which follows from the general results of S.M. Nikol'skii (see [15]). According to this relation for any function $f\in L_2(\mathbb{T}^n)$ the following equality holds

$$e_M(f)_{L_2(\mathbb{T}^n)}=\inf_{\Omega_M}\sup_{P\in\mathcal{L}^\perp,\|P\|_{L_2(\mathbb{T}^n)}\leq 1}\left|\int_{\mathbb{T}^n}f(\mathbf{x})\,P(\mathbf{x})d\mathbf{x}\right|, \tag{14}$$

where $\mathcal{L}$ is a linear span of a system of functions $\{e^{i(\mathbf{k},\mathbf{x})}\}_{\mathbf{k}\in\Omega_M}$.

We consider the function

$$f(\mathbf{x})=m^{-\sum_{j\in D'\setminus\{j_1\}}1/\tau_j}\sum_{m\leq\left(\gamma',\mathbf{s_0}\right)<m+n}\prod_{j=1}^n 2^{-(\alpha_j+1-1/p_j)s_j^0}\sum_{\mathbf{k}\in\rho(\mathbf{s_0})}e^{i(\mathbf{k},\mathbf{x})},$$

where $D'=\{j\in D:\ 2<\tau_j\}\cup\{j_1\}$, $\mathbf{s_0}=(s_1^0,\ \ldots,\ s_n^0)$, $s_j^0=s_j$ at $j\in D'$ and $s_j^0=0$ at $j\notin D'$.

In the paper [16] it was proved that the function $C_{12}f(\mathbf{x})$ belongs to the class $B_{\mathbf{p}}^{\alpha\tau}(\mathbb{T}^n)$.

Let us construct the function $P(\mathbf{x})$ satisfying the condition (14).

Let

$$u(\mathbf{x}) = \sum_{(\gamma',\mathbf{s_0})\leq m}\ \sum_{\mathbf{k}\in\rho(\mathbf{s_0})} e^{i(\mathbf{k},\mathbf{x})}, \tag{15}$$

and $\Omega_M$ be an arbitrary collection of integer vectors $\mathbf{k} = (k_1, \ldots, k_n)$.

We denote by

$$v(\mathbf{x}) = \sum_{(\gamma',\mathbf{s_0})\leq m}\ \sum_{\mathbf{k}\in\rho(\mathbf{s_0})\cap\Omega_M} e^{i(\mathbf{k},\mathbf{x})}$$

the function, containing only those terms of (15), for which $\mathbf{k} \in \Omega_M$. By Minkowski's inequality and Parseval's equality for function $w(\mathbf{x}) = u(\mathbf{x}) - v(\mathbf{x})$ we have

$$\|\omega\|_{L_2(\mathbb{T}^n)} \leq C_{13}M^{1/2}.$$

We consider the function $P(\mathbf{x}) = C_{13}^{-1}M^{-1/2}w(\mathbf{x})$, then $\|P\|_{L_2(\mathbb{T}^n)} \leq 1$. Since the function $w(\mathbf{x}) = u(\mathbf{x}) - v(\mathbf{x})$ does not contain the harmonics from $\Omega_M$, then function $P \in \mathcal{L}^{\perp}$. Thus, the function $P(\mathbf{x})$ satisfies the conditions from (14).

According to (14) and by Lemma 5 we obtain

$$e_M(f)_{L_2(\mathbb{T}^n)} \geq C_{14}M^{-1/2}\left|\int_{\mathbb{T}^n} f(\mathbf{x})\,\omega(\mathbf{x})d\mathbf{x}\right| \geq$$

$$\geq C_{14}M^{-1/2}m^{-\sum_{j\in D'\setminus\{j_1\}}1/\tau_j}\sum_{(\gamma',\mathbf{s_0})=m}\prod_{j=1}^{n}2^{-(\alpha_j+1-1/p_j)s_j^0}\sum_{\mathbf{k}\in\rho(\mathbf{s_0})}1 =$$

$$\geq C_{14}M^{-1/2}m^{-\sum_{j\in D'\setminus\{j_1\}}1/\tau_j}\sum_{(\gamma',\mathbf{s_0})=m}\prod_{j=1}^{n}2^{-(\alpha_j-1/p_j)s_j^0} =$$

$$= C_{14}M^{-1/2}m^{-\sum_{j\in D'\setminus\{j_1\}}1/\tau_j}\left\|\left\{2^{-\left(\alpha_{j_1}-1/p_{j_1}\right)(\mathbf{1},\mathbf{s})}\right\}_{\aleph^{|D|}(\mathbf{1},\mathbf{s})}\right\|_{l_1} \asymp$$

$$\asymp M^{-1/2}m^{-\sum_{j\in D'\setminus\{j_1\}}1/\tau_j}\cdot 2^{-\left(\alpha_{j_1}-1/p_{j_1}\right)m}m^{(|D|-1)}, \tag{16}$$

where $\bar{\mathbf{s}} = (s_{j_1}, \ldots, s_{j_{|D|}})$.

Taking into account that $M \asymp 2^m m^{(|D|-1)}$ from (16) we have

$$e_M(f)_{L_2(\mathbb{T}^n)} \geq C_{15}2^{-\left(\alpha_{j_1}-1/p_{j_1}+1/2\right)m}m^{\sum_{j\in D\setminus\{j_1\}}(1/2-1/\tau_j)_+} =$$

$$= C_{16}M^{-\left(\alpha_{j_1}-1/p_{j_1}+1/2\right)}(\log M)^{(|D|-1)\left(\alpha_{j_1}-1/p_{j_1}+1/2\right)+\sum_{j\in D\setminus\{j_1\}}(1/2-1/\tau_j)_+}. \tag{17}$$

By (17) lower estimate in (4) follows.

*Remark.* Note, that for $\mathbf{p} = (p, \ldots, p)$, $\tau = (\tau, \ldots, \tau)$ and $\mathbf{q} = \theta = (q, \ldots, q)$ the statement of the theorem coincides with the corresponding result of A.S. Romanyuk [7].

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Қ.А. Бекмағанбетов, Қ.Е. Кервенев, Е. Төлеуғазы

# Анизотропты Лоренц кеңістігіндегі метрикасындағы аралас метрикалы Никольский-Бесов класындағы тригонометриялық көлденеңнің реті

Тригонометриялық көлденең ұғымын бірөлшемді жағдайда алғаш рет Р.С. Исмагилов енгізді және үзіліссіз функциялар кеңістігінде бірқатар кластар үшін оларға бағалаулар белгіледі. Көп айнымалы функциялар үшін $L_q$ кеңістігінде Соболевтің $W_p^r$, Никольскийдің $H_p^r$ кластарындағы тригонометриялық көлденеңдер үшін дәл бағалауларды Э.С.Белинский, В.Е. Майоров, Ю. Маковоз, Г.Г. Магарил-Ильяев, В.Н. Темляков анықтады. Бұл есепті $B_{pq}^{\mathbf{r}}$ Бесов класы үшін А.С. Романюк, Д.Б. Базарханов зерттеді. Анизотропты Никольский-Бесов $B_{\mathbf{pr}}^{\alpha\tau}(\mathbb{T}^n)$ класы үшін тригонометриялық көлденең анизотропты Лоренц кеңістіктері метрикасында К.А. Бекмаганбетов және Е. Төлеуғазымен табылды.

*Кілт сөздер:* тригонометриялық көлденең, анизотропты Лоренц кеңістіктері, аралас метрикалы Никольский-Бесов класы.

К.А. Бекмаганбетов, К.Е. Кервенев, Е. Толеугазы

# Порядок тригонометрического поперечника класса Никольского-Бесова со смешанной метрикой в метрике анизотропного пространства Лоренца

Понятие тригонометрического поперечника в одномерном случае впервые введено Р.С. Исмагиловым и им были установлены оценки для некоторых классов в пространстве непрерывных функций. Для функций многих переменных точные порядки тригонометрических поперечников класса Соболева $W_p^r$, Никольского $H_p^r$ в пространстве $L_q$ установлены Э.С.Белинским, В.Е. Майоровым, Ю. Маковозом, Г.Г. Магарил-Ильяевым, В.Н. Темляковым. Эта задача для класса Бесова $B_{pq}^{\mathbf{r}}$ исследована А.С. Романюком и Д.Б. Базархановым. Тригонометрический поперечник для анизотропного класса Никольского-Бесова $B_{\mathbf{p}r}^{\alpha\tau}(\mathbb{T}^n)$ в метрике анизотропных пространств Лоренца $L_{\mathbf{q}\theta}(\mathbb{T}^n)$ был найден К.А. Бекмаганбетовым и Е. Толеугазы.

*Ключевые слова:* тригонометрический поперечник, анизотропные пространства Лоренца, класс Никольского-Бесова со смешанной метрикой.

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# One approach to solve a nonlinear boundary value problem for the Fredholm integro-differential equation

A quasilinear boundary value problem for a Fredholm integro-differential equation is considered. The interval is divided into $N$ parts and the values of the solution to the equation at the left end points of the subintervals are introduced as additional parameters. New unknown functions are introduced on the subintervals and a special Cauchy problem with parameters is solved with respect to a system of such functions. By means of the solution to this problem, a new general solution to the quasilinear Fredholm integro-differential equation is constructed. The conditions of the existence of a unique new general solution to the equation under consideration are obtained. A new general solution is used to create a system of nonlinear algebraic equations in parameters introduced. The conditions for the existence of a unique solution to this system are established. This ensures the existence of a unique solution to original problem

*Keywords:* quasilinear Fredholm integro-differential equation, quasilinear boundary value problem, a new general solution, iterative process.

*Introduction*

Integro-differential equations (IDEs) are often encountered in the applications as mathematical models of real processes [1–6] The solvability of different problems for IDEs and methods for solving them have been studied by many authors [1, 4–20] General solutions play an important role in investigating qualitative properties of problems for IDEs and solving them. However, the classical general solution exists not for all Fredholm integro-differential equations (FIDEs) (see [7, 10]). Therefore, a new concept of general solution to FIDE is proposed in [11]. Employing parametrization's method [21] and choosing a regular partition $\Delta_N$ of the interval $[0,T]$ (see [9, 10]), a $\Delta_N$ general solution $x(\Delta_N,t,\lambda)$ to the linear FIDE is introduced. In contradistinction the classical general solution, $x(\Delta_N,t,\lambda)$ exists for all linear FIDEs and depends on a parameter $\lambda \in R^{nN}$. The paper [22] introduces the concept of the $\Delta_N$ general solution to a nonlinear ordinary differential equation. In [12], the concept of the $\Delta_N$ general solution is extended to FIDEs with nonlinear differential parts. The use of such a solution allows one to reduce a nonlinear boundary value problem (BVP) to a system of nonlinear algebraic equations in parameters $\lambda_r$, $r=\overline{1,N}$.

We consider the quasilinear FIDE

$$\frac{dx}{dt} = A(t)x + \sum_{k=1}^{m} \varphi_k(t) \int_0^T \psi_k(\tau)x(\tau)d\tau + f_0(t) + \varepsilon f(t,x), \quad t\in[0,T], \quad x\in\mathbb{R}^n, \tag{1}$$

subject to the boundary condition

$$Bx(0) + Cx(T) = d, \quad d\in\mathbb{R}^n, \tag{2}$$

where $\varepsilon > 0$, the $n\times n$ matrices $A(t)$, $\varphi_k(t)$, $\psi_k(\tau)$, $k=\overline{1,m}$, and the $n$ vector $f_0(t)$ are continuous on $[0,T]$, $f:[0,T]\times R^n \to R^n$ is continuous, $\|x\| = \max\limits_{i=\overline{1,n}} |x_i|$.

The aim of this paper is to construct the $\Delta_N$ general solution to equation (1) by using analogous solution to a linear FIDE and solve BVP (1), (2).

Let us denote by $C\big([0,T],\ R^n\big)$ the space of all continuous functions $x:[0,T]\to R^n$ with the norm $\|x\|_1 = \max\limits_{t\in[0,T]} \|x(t)\|$. A solution to problem (1), (2) is a continuously differentiable on $[0,T]$ function $x(t)$ satisfying equation (1) and boundary condition (2). Here and below in the article, we assume that the functions observed at the end-points of the intervals have one-sided derivatives.

*1 The $\Delta_N$ general solution to equation (1)*

Let $\Delta_N$ be a partition of the interval $[0,T]$ with the points: $t_0 = 0 < t_1 < \ldots < t_N = T$.

We introduce the space $C\big([0,T],\Delta_N,R^{nN}\big)$ consisting of all function systems $x[t] = \big(x_1(t), x_2(t),\ldots,x_N(t)\big)$, where functions $x_r : [t_{r-1},t_r) \to R^n$, $r = \overline{1,N}$, are continuous and have finite left-sided limits $\lim\limits_{t\to t_r-0} x_r(t)$, with the norm $\big\|x[\cdot]\big\|_2 = \max\limits_{r=\overline{1,N}} \sup\limits_{t\in[t_{r-1},t_r)} \|x_r(t)\|$.

First, we set $\varepsilon = 0$ in equation (1) and consider the linear FIDE

$$\frac{dy}{dt} = A(t)y + \sum_{k=1}^{m} \varphi_k(t)\int_0^T \psi_k(\tau)y(\tau)d\tau + f_0(t), \quad t\in[0,T], \quad y\in\mathbb{R}^n. \tag{3}$$

Applying parametrization's method (see [10; 345]) to equation (3), for the partition $\Delta_N$, we get the special Cauchy problem for the system of IDEs with parameters

$$\frac{dv_r}{dt} = A(t)(v_r+\lambda_r) + \sum_{k=1}^{m}\varphi_k(t)\sum_{j=1}^{N}\int_{t_{j-1}}^{t_j}\psi_k(\tau)[v_j(\tau)+\lambda_j]d\tau + f_0(t), \quad t\in[t_{r-1},t_r), \tag{4}$$

$$v_r(t_{r-1}) = 0, \quad r = \overline{1,N}. \tag{5}$$

A solution to problem (4), (5) for a fixed parameter $\lambda = \lambda^* = (\lambda_1^*,\lambda_2^*,\ldots,\lambda_N^*) \in R^{nN}$ is a function system $v[t,\lambda^*] = \big(v_1(t,\lambda^*), v_2(t,\lambda^*),\ldots,v_N(t,\lambda^*)\big) \in C\big([0,T],\Delta_N,R^{nN}\big)$, where $v_r(t,\lambda^*)$, $r=\overline{1,N}$, are continuously differentiable with respect to $t$ on their domains, satisfy the system (4) for $\lambda_r = \lambda_r^*$, $r=\overline{1,N}$, and initial conditions (5).

We construct the $nm\times nm$ matrix $G(\Delta_N) = \big(G_{p,k}(\Delta_N)\big)$ with the elements

$$G_{p,k}(\Delta_N) = \sum_{r=1}^{N}\int\limits_{t_{r-1}}^{t_r}\psi_p(\tau)X_r(\tau)\int\limits_{t_{r-1}}^{\tau}X_r^{-1}(\tau_1)\varphi_k(\tau_1)d\tau_1 d\tau, \quad p,k=\overline{1,m},$$

where $X_r(t)$ is the fundamental matrix of differential equation $\dfrac{dx}{dt} = A(t)x$ on the interval $[t_{r-1},t_r]$.

Assume that the matrix $\big[I - G(\Delta_N)\big]$ is invertible and its inverse is represented in the form

$$\big[I-G(\Delta_N)\big]^{-1} = \big(\mathcal{R}_{k,p}(\Delta_N)\big), \qquad k,p=\overline{1,m},$$

where $I$ is the identity matrix of dimension $nm$, $\mathcal{R}_{k,p}(\Delta_N)$ are square matrices of dimension $n$.

The invertibility of the matrix $I - G(\Delta_N)$ provides the existence of a function system $v[t,\lambda] = \big(v_1(t,\lambda), v_2(t,\lambda),\ldots,v_N(t,\lambda)\big) \in C\big([0,T],\Delta_N,R^{nN}\big)$, a unique solution to the special Cauchy problem (4), (5) for any $\lambda = (\lambda_1,\lambda_2,\ldots,\lambda_N)\in R^{nN}$ and $f_0(t)\in C\big([0,T],R^n\big)$. Moreover, the following inequality is valid

$$\big\|v[\cdot,\lambda]\big\|_2 \le \chi\big\|F_0[\cdot,\lambda]\big\|_2,$$

where $\chi$ is a constant independent of $\lambda\in R^{nN}$ and $f_0(t)\in C\big([0,T],R^n\big)$, and $F_0[t,\lambda] = \big(F_{0,1}(t,\lambda), F_{0,2}(t,\lambda),\ldots,F_{0,N}(t,\lambda)\big) \in C\big([0,T],\Delta_N,R^{nN}\big)$, with elements $F_{0,r}(t,\ \lambda) = A(t)\lambda_r + \sum\limits_{k=1}^{m}\varphi_k(t)\sum\limits_{j=1}^{N}\int_{t_{j-1}}^{t_j}\psi_k(\tau)d\tau\lambda_j + f_0(t), \quad t\in[t_{r-1},t_r)\ r=\overline{1,N}$.

The number $\chi$ is called a well-posedness constant of the special Cauchy problem (4), (5). Since $I - G(\Delta_N)$ is invertible then, by results obtained in [11], there exists a unique $\Delta_N$ general solution $y(\Delta_N,t,\lambda)$ to equation (3) and

$$y(\Delta_N,t,\lambda) = \lambda_r + \sum_{j=1}^{N} d_{r,j}(\Delta_N,t)\lambda_j + b_r(\Delta_N,t), \quad t\in[t_{r-1},t_r), \quad r=\overline{1,N},$$

$$y(\Delta_N,T,\lambda) = \lambda_N + \sum_{j=1}^{N} d_{N,j}(\Delta_N,T)\lambda_j + b_N(\Delta_N,T),$$

with the following coefficients and right-hand sides

$$d_{r,j}(\Delta_N,t)=X_r(t)\int\limits_{t_{r-1}}^{t}X_r^{-1}(\tau)\sum_{k=1}^{m}\varphi_k(\tau)\Big[\sum_{p=1}^{m}\mathcal{R}_{k,p}(\Delta_N)V_{p,j}(\Delta_N)+$$

$$+\int\limits_{t_{j-1}}^{t_j}\psi_k(\tau_1)d\tau_1\Big]d\tau,\qquad t\in[t_{r-1},t_r],\quad j\neq r,\quad r,j=\overline{1,N},$$

$$d_{r,r}(\Delta_N,t)=X_r(t)\int\limits_{t_{r-1}}^{t}X_r^{-1}(\tau)\Big\{\sum_{k=1}^{m}\varphi_k(\tau)\Big[\sum_{p=1}^{m}\mathcal{R}_{k,p}(\Delta_N)V_{p,r}(\Delta_N)+\int\limits_{t_{r-1}}^{t_r}\psi_k(\tau_1)d\tau_1\Big]+A(\tau)\Big\}d\tau,$$

$$b_r(\Delta_N,t)=X_r(t)\int\limits_{t_{r-1}}^{t}X_r^{-1}(\tau)\Big[\sum_{k=1}^{m}\varphi_k(\tau)\sum_{p=1}^{m}\mathcal{R}_{k,p}(\Delta_N)g_p(\Delta_N,f_0)+f_0(\tau)\Big]d\tau,\quad r=\overline{1,N},$$

$$V_{p,r}(\Delta_N)=\int\limits_{t_{r-1}}^{t_r}\psi_p(\tau)X_r(\tau)\int\limits_{t_{r-1}}^{\tau}X_r^{-1}(\tau_1)A(\tau_1)d\tau_1d\tau+$$

$$+\sum_{j=1}^{N}\sum_{k=1}^{m}\int\limits_{t_{j-1}}^{t_j}\psi_p(\tau)X_r(\tau)\int\limits_{t_{j-1}}^{\tau}X_r^{-1}(\tau_1)\varphi_k(\tau_1)d\tau_1d\tau\int\limits_{t_{r-1}}^{t_r}\psi_k(\tau_2)d\tau_2,$$

$$g_p(\Delta_N,f)=\sum_{r=1}^{N}\int\limits_{t_{r-1}}^{t_r}\psi_p(\tau)X_r(\tau)\int\limits_{t_{r-1}}^{\tau}X_r^{-1}(\tau_1)f(\tau_1)d\tau_1d\tau.$$

Given a vector $\lambda^{(0)}=\left(\lambda_1^{(0)},\lambda_2^{(0)},\dots,\lambda_N^{(0)}\right)\in R^{nN}$ and numbers $\rho_\lambda>0$, $\rho>\rho_\lambda$, $\rho_u=\rho-\rho_\lambda$, we choose the piecewise continuous on $[0,T]$ function $y^{(0)}(t)=y\big(\Delta_N,t,\lambda^{(0)}\big)$, the function system $v^{(0)}[t]=\big(v_1^{(0)}(t),v_2^{(0)}(t),\dots,v_N^{(0)}(t)\big)$ with elements $v_r^{(0)}(t)=y^{(0)}(t)-\lambda_r^{(0)}$, $t\in[t_{r-1},t_r)$, $r=\overline{1,N}$, and compose the following sets

$G^0(\rho)=\Big\{(t,x):t\in[0,T],\big\|x-y^{(0)}(t)\big\|<\rho\Big\}$,

$S\big(\lambda^{(0)},\rho_\lambda\big)=\Big\{\lambda=(\lambda_1,\lambda_2,\dots,\lambda_N)\in R^{nN}:\big\|\lambda_r-\lambda_r^{(0)}\big\|<\rho_\lambda,\quad r=\overline{1,N}\Big\}$,

$S\big(v^{(0)}[t],\rho_u\big)=\Big\{u[t]\in C\big([0,T],\Delta_N,R^{nN}\big):\|u[\cdot]-v^{(0)}[\cdot]\|_2<\rho_u\Big\}$,

$G_p^0(\rho)=\Big\{(t,x):t\in[t_{p-1},t_p),\|x-y^{(0)}(t)\|<\rho\Big\},\quad p=\overline{1,N-1}$,

$G_N^0(\rho)=\Big\{(t,x):t\in[t_{N-1},t_N],\|x-y^{(0)}(t)\|<\rho\Big\}$, and $G^0(\Delta_N,\rho)=\bigcup\limits_{r=1}^{N}G_r^0(\rho)$.

In order to construct the $\Delta_N$ general solution to equation (1), we employ again the parametrization's method.

If a function $x(t)$ satisfies equation (1) and $(t,x(t))\in G^0(\Delta_N,\rho)$, then the functions $x_r(t)$, $r=\overline{1,N}$, as the restrictions of $x(t)$ to $[t_{r-1},t_r)$, satisfy the system of nonlinear IDEs

$$\frac{dx_r}{dt}=A(t)x_r+\sum_{k=1}^{m}\varphi_k(t)\sum_{j=1}^{N}\int_{t_{j-1}}^{t_j}\psi_k(\tau)x_j(\tau)d\tau+f_0(t)+\varepsilon f(t,x_r),\ t\in[t_{r-1},t_r),$$

and $(t,x_r(t))\in G_r^0(\rho)$, $r=\overline{1,N}$. Introducing the parameters $\lambda_r\hat{=}x_r(t_{r-1})$ and making the substitutions $u_r(t)=x_r(t)-\lambda_r$, $t\in[t_{r-1},t_r)$, $r=\overline{1,N}$, we obtain the system of nonlinear IDEs with parameters

$$\frac{du_r}{dt}=A(t)(u_r+\lambda_r)+\sum_{k=1}^{m}\varphi_k(t)\sum_{j=1}^{N}\int_{t_{j-1}}^{t_j}\psi_k(\tau)[u_j(\tau)+\lambda_j]d\tau+f_0(t)+\varepsilon f(t,u_r+\lambda_r),\quad t\in[t_{r-1},t_r),\quad(6)$$

subject to the initial conditions

$$u_r(t_{r-1}) = 0, \quad r = \overline{1,N}. \tag{7}$$

Problem (6), (7) is the special Cauchy problem for the system of nonlinear IDEs with parameters.

We represent problem (6), (7) as an operator equation and apply an iterative process for finding its solution. Set $X = \Big\{u[t] = \big(u_1(t), u_2(t), \dots, u_N(t)\big) \in C\big([0,T], \Delta_N, R^{nN}\big) : u_r(t_{r-1}) = 0,\ r = \overline{1,N}\Big\}$, $Y = C\big([0,T], \Delta_N, R^{nN}\big)$, and introduce the linear operator $H : X \to Y$ in the following way:

$$Hu[t] = \Big(w_1^{(1)}(t), w_2^{(1)}(t), \dots, w_N^{(1)}(t)\Big),$$

with $w_r^{(1)}(t) = \dot{u}_r(t) - A(t)u_r - \sum\limits_{k=1}^{m} \varphi_k(t) \sum\limits_{j=1}^{N} \int_{t_{j-1}}^{t_j} \psi_k(\tau) u_j(\tau) d\tau, \quad t \in [t_{r-1}, t_r), \quad r = \overline{1,N}.$

The domain of $H$ is $D(H) = \Big\{u[t] = \big(u_1(t), u_2(t), \dots, u_N(t)\big) \in X$, where $u_r(t)$ is continuously differentiable on $[t_{r-1}, t_r)$, $r = \overline{1,N}\Big\}$. It is easily seen that $H$ is a closed unbounded linear operator.

Now, we can write down the special Cauchy problem (6), (7) as a nonlinear operator equation

$$Hu[t] = \varepsilon F\big(u[t], \lambda\big) + F_0[t, \lambda], \tag{8}$$

with $F\big(u[t], \lambda\big) = \big(w_1^{(2)}(t), w_2^{(2)}(t), \dots, w_N^{(2)}(t)\big), \quad w_r^{(2)}(t) = f(t, u_r(t) + \lambda_r), \quad t \in [t_{r-1}, t_r),\ r = \overline{1,N}.$

Let $L(Y,X)$ be the space of linear bounded operators $\Lambda : Y \to X$ with the induced norm. Our assumption that the special Cauchy problem (4), (5) is well-posed with the constant $\chi$ leads to the invertibility of the operator $H : X \to Y$ and the estimate $\|H^{-1}\|_{L(Y,X)} \le \chi$.

*Theorem 1. Let the special Cauchy problem (4), (5) be well-posed with a constant $\chi$ and the following inequalities be valid:*
(i) $\big\|f(t,x') - f(t,x'')\big\| \le L\|x' - x''\|$, *$L$ is a constant,* $(t,x'), (t,x'') \in G^0(\rho)$;
(ii) $q_\varepsilon = \varepsilon\chi L < 1$;
(iii) $\dfrac{1}{1-q_\varepsilon}\varepsilon\chi \max\limits_{r=\overline{1,N}} \sup\limits_{t\in[t_{r-1},t_r)} \big\|f\big(t, v_r(t,\lambda) + \lambda_r\big)\big\| < \rho_u$ *for all* $\lambda \in S(\lambda^{(0)}, \rho_\lambda)$.

*Then for any $\lambda \in S(\lambda^{(0)}, \rho_\lambda)$, there exists a unique function system $u[t,\lambda,\varepsilon] = (u_1(t,\lambda,\varepsilon), u_2(t,\lambda,\varepsilon), \dots, u_N(t,\lambda,\varepsilon))$, the solution to the special Cauchy problem (6), (7) belonging to $S\big(v^{(0)}[t], \rho_u\big)$, and the following inequality is true*

$$\big\|u[\cdot,\lambda,\varepsilon] - v[\cdot,\lambda]\big\|_2 \le \frac{1}{1-q_\varepsilon}\varepsilon\chi \max_{r=\overline{1,N}} \sup_{t\in[t_{r-1},t_r)} \big\|f\big(t, v_r(t,\lambda) + \lambda_r\big)\big\|. \tag{9}$$

*Proof.* Since the operator $H$ has a bounded inverse, equation (8) is equivalent to the next operator equation:

$$u[t] = \varepsilon H^{-1} F\big(u[t], \lambda\big) + H^{-1} F_0[t, \lambda]. \tag{10}$$

For any fixed $\lambda \in S(\lambda^{(0)}, \rho_\lambda)$, the solution to equation (10) we find by the iterative process

$$u^{(0)}[t,\lambda,\varepsilon] = v[t,\lambda],$$

$$u^{(\nu+1)}[t,\lambda,\varepsilon] = \varepsilon H^{-1} F\big(u^{(\nu)}[t,\lambda,\varepsilon], \lambda\big) + H^{-1} F_0[t,\lambda], \quad \nu = 0, 1, 2, \dots, \tag{11}$$

Using our assumptions, we obtain the following inequalities:

$$\big\|u^{(1)}[\cdot,\lambda,\varepsilon] - v[\cdot,\lambda]\big\|_2 = \varepsilon\big\|H^{-1}F\big(v[\cdot,\lambda],\lambda\big)\big\|_2 \le \varepsilon\chi \max_{r=\overline{1,N}} \sup_{t\in[t_{r-1},t_r)} \big\|f\big(t, v_r(t,\lambda) + \lambda_r\big)\big\|, \tag{12}$$

$$\Big\|u^{(\nu+1)}[\cdot,\lambda,\varepsilon] - u^{(\nu)}[\cdot,\lambda,\varepsilon]\Big\|_2 \le \varepsilon\big\|H^{-1}\big\|_{L(Y,X)} \Big\|F\big(u^{(\nu)}[\cdot,\lambda,\varepsilon]\big) - F\big(u^{(\nu-1)}[\cdot,\lambda,\varepsilon]\big)\Big\|_2 \le$$

$$\le \varepsilon\chi \max_{r=\overline{1,N}} \sup_{t\in[t_{r-1},t_r)} \Big\|f\big(t, u_r^{(\nu)}(t,\lambda,\varepsilon) + \lambda_r\big) - f\big(t, u_r^{(\nu-1)}(t,\lambda,\varepsilon) + \lambda_r\big)\Big\| \le \tag{13}$$

$$\leq \varepsilon\chi L\big\|u^{(\nu)}[\cdot,\lambda,\varepsilon]-u^{(\nu-1)}[\cdot,\lambda,\varepsilon]\big\|_2, \quad \nu=1,2,\ldots,$$

$$\big\|u^{(\nu+1)}[\cdot,\lambda,\varepsilon]-v[\cdot,\lambda]\big\|_2<\frac{1}{1-q_\varepsilon}\varepsilon\chi\max_{r=\overline{1,N}}\sup_{t\in[t_{r-1},t_r)}\big\|f\big(t,v_r(t,\lambda)+\lambda_r\big)\big\|. \tag{14}$$

The inequalities (12)–(14) and condition (iii) of Theorem 1 provide the convergence of the iterative process (11) to the function system $u[t,\lambda,\varepsilon]$, a unique solution to equation (8) in $S(v[t,\lambda],\rho_u)$, and validity of estimate (9). □

*Definition 1. Let a function system* $u[t,\lambda,\varepsilon]=\big(u_1(t,\lambda,\varepsilon),u_2(t,\lambda,\varepsilon),\ldots,u_N(t,\lambda,\varepsilon)\big)\in S(v[t,\lambda],\rho_u)$ *be a unique solution to the special Cauchy problem (6), (7) with parameters* $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_N)\in S(\lambda^{(0)},\rho_\lambda)$. *Then the function* $x(\Delta_N,t,\lambda,\varepsilon)$ *given by the equalities:* $x(\Delta_N,t,\lambda,\varepsilon)=\lambda_r+u_r(t,\lambda,\varepsilon)$ *for* $t\in[t_{r-1},t_r)$, $r=\overline{1,N}$, *and* $x(\Delta_N,T,\lambda,\varepsilon)=\lambda_N+\lim\limits_{t\to T-0}u_N(t,\lambda,\varepsilon)$, *is called a* $\Delta_N$ *general solution to equation (1) in* $G^0(\Delta_N,\rho)$.

Definition 1 and Theorem 1 imply the following assertion.

*Theorem 2. Under conditions of Theorem 1, there exists a function* $x(\Delta_N,t,\lambda,\varepsilon)$, *which is a unique* $\Delta_N$ *general solution to equation (1) in* $G^0(\Delta_N,\rho)$, *and this function can be represented in the form*

$$x(\Delta_N,t,\lambda,\varepsilon)=y(\Delta_N,t,\lambda)+\Delta x(\Delta_N,t,\lambda,\varepsilon),$$

where the function $\Delta x(\Delta_N,t,\lambda,\varepsilon)$ is compiled by the equalities $\Delta x(\Delta_N,t,\lambda,\varepsilon)=u_r(t,\lambda,\varepsilon)-v_r(t,\lambda)$, for $t\in[t_{r-1},t_r)$, $r=\overline{1,N}$, $\Delta x(\Delta_N,T,\lambda,\varepsilon)=\lim\limits_{t\to T-0}u_N(t,\lambda,\varepsilon)-\lim\limits_{t\to T-0}v_N(t,\lambda)$. Moreover, the following estimate is valid

$$\sup_{t\in[0,T]}\|\Delta x(\Delta_N,t,\lambda,\varepsilon)\|\leq\frac{1}{1-q_\varepsilon}\varepsilon\chi\max_{r=\overline{1,N}}\sup_{t\in[t_{r-1},t_r)}\big\|f\big(t,v_r(t,\lambda)+\lambda_r\big)\big\|.$$

## *2 The solvability of problem (1)–(2)*

In this Section, we investigate the solvability of quasilinear BVP (1)–(2). We first consider a linear BVP for equation (3) with the boundary condition (2).

Substituting the $\Delta_N$ general solution to equation (3) into the boundary condition (2) and the continuity conditions at the interior points of the partition, we obtain the system of linear algebraic equations in parameters

$$B\lambda_1+C\lambda_N+C\sum_{j=1}^N d_{N,j}(\Delta_N,T)\lambda_j=d-Cb_N(\Delta_N,T), \tag{15}$$

$$\lambda_p+\sum_{j=1}^N d_{p,j}(\Delta_N,t_p)\lambda_j-\lambda_{p+1}=-b_p(\Delta_N,t_p), \quad p=\overline{1,N-1}. \tag{16}$$

We rewrite equations (15), (16) in the form

$$Q_*(\Delta_N)\lambda=-F_*(\Delta_N).$$

In accordance with Theorem 2.2 in [10], the invertibility of the matrix $Q_*(\Delta_N):R^{nN}\to R^{nN}$ is equivalent to the unique solvability of linear BVP (3), (2).

Now, we study the solvability of quasilinear BVP (1), (2). If $x(t)$ is a solution to equation (1), and $x[t]=(x_1(t),x_2(t),...,x_N(t))$ is a function system of its restrictions to the subintervals $[t_{r-1},t_r)$, $r=\overline{1,N}$, then the equations

$$\lim_{t\to t_p-0}x_p(t)=x_{p+1}(t_p), \qquad p=\overline{1,N-1}, \tag{17}$$

hold. Equations (17) are the continuity conditions for solutions to equation (1) at the interior points of partition $\Delta_N$.

Let $x(\Delta_N,t,\lambda,\varepsilon)$ be a $\Delta_N$ general solution to equation (1) in $G^0(\Delta_N,\rho)$. Substituting the corresponding expressions of $x(\Delta_N,t,\lambda,\varepsilon)$ into boundary condition (2) and continuity conditions (17), we get the system of nonlinear algebraic equations

$$B\lambda_1+C\lambda_N+C\sum_{j=1}^N d_{N,j}(\Delta_N,T)\lambda_j+C\Delta x(\Delta_N,T,\lambda,\varepsilon)=d-Cb_N(\Delta_N,T), \tag{18}$$

$$\lambda_p + \sum_{j=1}^{N} d_{p,j}(\Delta_N, t_p)\lambda_j - \lambda_{p+1} + \Delta x(\Delta_N, t_p, \lambda, \varepsilon) = -b_p(\Delta_N, t_p), \quad p = \overline{1, N-1}. \tag{19}$$

We rewrite system (18), (19) in the form:

$$Q_*(\Delta_N)\lambda = -F_*(\Delta_N) - \Delta Q_*(\Delta_N, \lambda, \varepsilon), \tag{20}$$

where

$$\Delta Q_*(\Delta_N, \lambda, \varepsilon) = \begin{pmatrix} C\Delta x(\Delta_N, T, \lambda, \varepsilon) \\ \Delta x(\Delta_N, t_1, \lambda, \varepsilon) \\ \dots \\ \Delta x(\Delta_N, t_{N-1}, \lambda, \varepsilon) \end{pmatrix}.$$

As proved in Theorem 3.2 [12; 31] the solvability of problem (1), (2) is equivalent to that of system of nonlinear algebraic equations (20). The conditions of the solvability of (20) are established in the following statement.

*Theorem 3. Let the conditions of Theorem 1 are met and the following assumptions hold:*
(i) $Q_*(\Delta_N)$ *is invertible and* $\|[Q_*(\Delta_N)]^{-1}\| \le \gamma$;
(ii) $\sigma_\varepsilon = q_\varepsilon\Big(\dfrac{\chi}{1-q_\varepsilon}\big(\alpha + K_0 + \varepsilon L\big) + 1\Big) < 1$,
*where* $\alpha = \max\limits_{t\in[0,T]} \max\limits_{i=\overline{1,n}} \sum\limits_{j=1}^{n} \|a_{ij}(t)\|$, $K_0 = \sum\limits_{k=1}^{m} \max\limits_{t\in[0,T]} \|\varphi_k(t)\| \sum\limits_{j=1}^{N} \int_{t_{j-1}}^{t_j} \|\psi_k(\tau)\| d\tau$;
(iii) $\dfrac{1}{1-\sigma_\varepsilon} \cdot \dfrac{\varepsilon\chi}{1-q_\varepsilon}\gamma \max\big(1, \|C\|\big) \max\limits_{r=\overline{1,N}} \sup\limits_{t\in[t_{r-1},t_r)} \big\|f\big(t, v_r(t,\lambda^{(0)}) + \lambda_r^{(0)}\big)\big\| < \rho_\lambda$.

*Then system of nonlinear algebraic equations (20) has a unique solution* $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_N) \in S(\lambda^{(0)}, \rho_\lambda)$.

*Proof.* A solution to equation (20) is found by the iterative process

$$\lambda^{(0)} = [Q_*(\Delta_N)]^{-1} \cdot F_*(\Delta_N),$$

$$\lambda^{(\nu+1)} = -[Q_*(\Delta_N)]^{-1}\Big\{F_*(\Delta_N) + \Delta Q_*\big(\Delta_N, \lambda^{(\nu)}, \varepsilon\big)\Big\}. \tag{21}$$

Under conditions of Theorem the following inequalities hold:

$$\|\lambda^{(1)} - \lambda^{(0)}\| \le \gamma \cdot \Big\|\Delta Q_*(\Delta_N, \lambda^{(0)}, \varepsilon)\Big\| \le \gamma \cdot \max\big(1, \|C\|\big) \max_{r=\overline{1,N}} \Big\|\Delta x(\Delta_N, t_r, \lambda^{(0)}, \varepsilon)\Big\| \le$$
$$\le \gamma \cdot \max\big(1, \|C\|\big) \frac{\varepsilon\chi}{1-q_\varepsilon} \max_{r=\overline{1,N}} \sup_{t\in[t_{r-1},t_r)} \big\|f\big(t, v_r(t,\lambda^{(0)}) + \lambda_r^{(0)}\big)\big\|,$$

$$\|\lambda^{(\nu+1)} - \lambda^{(\nu)}\| \le q_\varepsilon\Big\{\frac{\chi}{1-q_\varepsilon}(\alpha + K_0 + \varepsilon L) + 1\Big\}\|\lambda^{(\nu)} - \lambda^{(\nu-1)}\|, \quad \nu = 1, 2, \dots,$$

$$\|\lambda^{(\nu+1)} - \lambda^{(0)}\| \le \frac{1}{1-\sigma_\varepsilon} \cdot \frac{\varepsilon\chi}{1-q_\varepsilon}\gamma \max\big(1, \|C\|\big) \max_{r=\overline{1,N}} \sup_{t\in[t_{r-1},t_r)} \big\|f\big(t, v_r(t,\lambda^{(0)}) + \lambda_r^{(0)}\big)\big\|.$$

Similarly to the proof of Theorem 1, the iterative process (21) converges to the vector $\lambda \in S(\lambda^{(0)}, \rho_\lambda)$, a unique solution to equation (20). □

From Theorem 1 and Theorem 3.2 in [12], we obtain

*Theorem 4. Let the conditions of Theorem 3 be fulfilled. Then quasilinear BVP (1), (2) has a unique solution* $x^*(t)$ *such that* $(t, x^*(t)) \in G^0(\Delta_N, \rho)$.

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## Фредгольм интегралдық-дифференциалдық теңдеуі үшін сызықты емес шеттік есепті шешудің бір тәсілі

Фредгольм интегралдық-дифференциалдық теңдеуі үшін квазисызықты шеттік есеп қарастырылды. Интервал $N$ бөлікке бөлінеді және қарастырылып отырған теңдеу шешімінің ішкі интервалдардың сол жақ шеткі нүктелеріндегі мәндері қосымша параметрлер ретінде енгізіледі. Ішкі интервалдарға белгісіз функциялар енгізіледі және осы функциялар жүйесіне қатысты параметрлері бар арнайы Коши есебі шешіледі. Арнайы Коши есебінің табылған шешімі арқылы квазисызықты Фредгольм интегралдық-дифференциалдық теңдеуінің жаңа жалпы шешімі құрылады. Қарастырылып отырған теңдеудің жалғыз жаңа шешімі бар болу шарттары алынған. Жаңа жалпы шешімнің көмегімен енгізілген параметрлерге қатысты сызықты емес алгебралық теңдеулер жүйесі құрылады. Осы жүйенің жалғыз шешімі бар болу шарттары тағайындалған, бұл шарттар квазисызықты шеттік есептің жалғыз шешімінің бар болуын қамтамасыз етеді.

*Кілт сөздер:* квазисызықты Фредгольм интегралдық-дифференциалдық теңдеуі, квазисызықты шеттік есеп, жаңа жалпы шешім, итерациялық процесс.

Д.С. Джумабаев, С.Т. Мынбаева

## Один подход к решению нелинейной краевой задачи для интегро-дифференциального уравнения Фредгольма

Рассмотрена квазилинейная краевая задача для интегро-дифференциального уравнения Фредгольма. Интервал поделен на $N$ частей, а значения решения рассматриваемого уравнения в левых конечных точках подинтервалов введены в качестве дополнительных параметров. На подинтервалах введены новые неизвестные функции и относительно этой системы функций решена специальная задача Коши с параметрами. Новое общее решение квазилинейного интегро-дифференциального уравнения Фредгольма построено через найденное решение специальной задачи Коши. Получены условия существования единственного нового общего решения рассматриваемого уравнения. С помощью нового общего решения составлена система нелинейных алгебраических уравнений относительно введенных параметров. Установлены условия существования единственного решения этой системы, обеспечивающие наличие единственного решения квазилинейной краевой задачи.

*Ключевые слова:* квазилинейное интегро-дифференциальное уравнение Фредгольма, квазилинейная краевая задача, новое общее решение, итерационный процесс.

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# Cohomology of simple modules for algebraic groups

In this paper, we consider questions related to the study of the cohomology of simple and simply connected algebraic groups with coefficients in simple modules. There are various calculating methods for them. One of the effective methods is to study the properties of the Lyndon–Hochschild–Serre spectral sequence with respect to the infinitesimal subgroup, the Frobenius kernel of a given algebraic group, and the properties of various cohomological sequences. We have studied the properties of various short exact and corresponding long exact cohomological sequences of modules over an algebraic group associated with simple modules with highest restricted weights. Some properties of the cohomology of the Frobenius kernel with coefficients in simple modules with higher restricted weights are described. We also studied the properties of the Lyndon–Hochschild–Serre spectral sequence on the first quadrant for simple modules with highest restricted weights. The limiting values of the points of the first quadrant of the spectral sequence are described. It is proved that for the simple, simply connected algebraic group $G$ over an algebraically closed field $k$ of characteristic $p > h$ with an irreducible root system $R$ and for a simple $G$-module $V$ with restricted highest weight, there is an isomorphism of $G$-modules

$$H^j(G, V) \cong Hom_G(k, H^j(G^1, V)^{(-1)}) \text{ for all } j \geq 0,$$

where $G^1$ is the Frobenius map kernel for $G$, $h$ is the Coxeter number of the root system $R$. This isomorphism allows us to reduce the calculation of the cohomology of group $G$ with coefficients in simple modules with higher restricted weights to the calculation of the corresponding cohomology of the Frobenius kernel $G^1$.

*Keywords:* algebraic group, Chevalley group, representation of Lie group, Frobenius kernel, simple module, cohomology, spectral sequence, exact sequence, restricted weight.

## *Introduction*

The cohomology of simple modules is known only for small degrees cohomology and for small groups. For example, the first degree cohomology of simple modules are completely calculated for $SL_2$ [1], $SL_3$ [2], $Sp_4$ [3], $G_2$, $p \geq 13$ [4]. Similar results for the second cohomology of simple modules were obtained for the following groups: $SL_2$ [5], $SL_3$ [6], $Sp_4$, $p > 7$ [7], and $G_2$, $p \geq 7$ [8]. The third degree cohomology of simple modules is described for all simple algebraic groups of rank two [9].

In this paper, we continued the research which started in [10], [11]. We studied the cohomology of simple, simply connected algebraic groups with an irreducible root system over an algebraically closed field of positive characteristic with coefficients in simple modules with restricted highest weights. Let $G$ be a simple, simply connected algebraic group over an algebraically closed field $k$ of characteristic $p > 0$ with an irreducible root system $R$, $\mathfrak{g}$ be a Lie algebra of the group $G$, and $G^1$ be the kernel of the Frobenius map $F_G : G \to G$. We will also apply the following standard notation:

$B$ is the Borel subgroup of $G$,
$T$ is the maximal torus of $G$,
$R_+$ is the set of positive roots,
$S = \{\alpha_1, \cdots, \alpha_l\}$ is the set of simple roots,
$\lambda_1, \cdots, \lambda_l$ are the fundamental weights,
$X(T)$ is the additive character group of $T$,
$X_+(T) = \{\lambda \in X(T) \,|\, \langle\lambda, \alpha^\vee\rangle \geq 0 \text{ для всех } \alpha \in S\}$ is the set of dominant weights,
$X_1(T) = \{\lambda \in X(T) \,|\, 0 \leq \langle\lambda, \alpha^\vee\rangle < p \text{ для всех } \alpha \in S\}$ is the set of restricted weights,
$s_{\alpha,rp} \cdot \lambda = \lambda - \langle\lambda + \rho, \alpha^\vee\rangle\alpha + rp\alpha$, $\alpha \in R_+$, $r \in \mathbb{Z}$ is the action of the affine Weyl group $W_p$ on $X(T)$,
$k_\lambda$ is the one-dimensional $B$-module,

$H^0(\lambda) = Ind_B^G(k_\lambda)$ is the $G$-module induced from the one-dimensional representation $k_\lambda$ of the Borel subgroup,

$L(\lambda)$ is the simple $G$-module with the highest weight $\lambda \in X_+(T)$.

For a rational $G$-module $L$, denote by $L^{(d)}$ the Frobenius twist of degree $d$. Thens, there is a unique $d > 0$ and a rational $G$-module $V$ such that $V^{(d)} = L$. Denote it by $L^{(-d)}$.

We say that $G$-module $L$ admits *a good filtration (or $H^0$-filtration)* if the filtration factors are isomorphic to the modules induced from the one-dimensional representations of the Borel subgroup of $G$. A rational module $L$ over an algebraic group $G$ is called *acyclic (or $G$-acyclic)* if $H^j(G, L) = 0$ for all $j > 0$. For a simple $G$-module $V$ with restricted highest weight, the following isomorphisms of $G$-modules were obtained before:

$$H^1(G, V) \cong Hom_G(k, H^1(G^1, V)^{(-1)}) \text{ (see [10, (2.4)])}$$

and

$$H^2(G, V) \cong Hom_G(k, H^2(G^1, V)^{(-1)}),\ p \geq 3h - 3,\ \text{(see [11, (3.2)])}.$$

In this paper, it is proved that a similar isomorphism is also hold in the general case. The main result is as follows

*Theorem 1. Let $G$ be the algebraic group with an irreducible root system over an algebraically closed field $k$ of characteristic $p > h$ and $V$ be a simple finite-dimensional $G$-module with restricted highest weight. Then*

$$H^j(G, V) \cong Hom_G(k, H^j(G^1, V)^{(-1)}) \text{ for all } j \geq 0.$$

To obtain the condition $p > h$ we use the following known facts:

• If $p > h$ and $\lambda = w \cdot 0 + p\nu$ for some $\nu \in X_+(T)$ and $w \in W$, then

$$H^i(G^1, H^0(\lambda))^{(-1)} \cong H^0(S^{(i-l(w))/2}(\mathbf{u}^*) \otimes k_\nu), \quad (1)$$

where $\mathfrak{u}$ is the maximal nilpotent subalgebra of the Lie algebra $\mathfrak{g}$ that corresponds to negative roots and $S(\mathfrak{u}^*)$ is the symmetric algebra of $\mathfrak{u}^*$ [12, p. 478], [13].

To apply formula (1) to calculate $H^i(G^1, H^0(w \cdot 0 + p\nu))^{(-1)}$, we will use the following character formula [12; 501]:

$$\chi(H^0(S^{(i-l(w))/2}(\mathbf{u}^*) \otimes k_\nu)) = \sum_{\mu \in X_+(T)} \sum_{w \in W} (-1)^{l(w)} P_{(i-l(w))/2}(w \cdot \mu - \nu)\chi(\mu), \quad (2)$$

where $P_{(i-l(w))/2}(w \cdot \mu - \nu)$ is the dimension of $(w \cdot \mu - \nu)$-weight subspace of $S^{(i-l(w))/2}(\mathfrak{u}^*)$.

• It is well known that the cohomology groups $H^m(G^1, H^0(\lambda))^{(-1)}$ $(\lambda \in X_1(T))$ as $G$-modules admit a good filtration [12, Lemma 4.5]. Recall that a more hard condition was used in [11], the completely reducibility of the $G$-module $H^1(G^1, V)^{(-1)}$.

• If $\lambda \in X_+(T)$ and $j > 0$ then $H^j(G, H^0(\lambda)) = 0$, that is, the induced module $H^0(\lambda)$ is $G$-acyclic [14, Corol. 3.4], [15, Lemma 2.1, (iii)], [16, II.4.13, (1)].

To prove Theorem 1, we will also use the properties of the Lyndon–Hochschild–Serre spectral sequence. For the short exact sequence of group schemes

$$1 \to G^1 \to G \to G/G^1 \to 1$$

and the $G$-module $V$, the following Lyndon–Hochschild–Serre spectral sequence holds [16, I.6.6.(3)]:

$$E_2^{nm} = H^n(G/G^1, H^m(G^1, V)) \Rightarrow H^{n+m}(G, V); \quad (3)$$

Let $V$ be a simple $G$-module with restricted highest weight. According to [17, Sec.1, p. 768],

$$H^n(G/G^1, H^m(G^1, V)) \cong H^n(G, H^m(G^1, V)^{(-1)}).$$

Hence,

$$E_2^{nm} \cong H^n(G, H^m(G^1, V)^{(-1)}). \quad (4)$$

If $E_\infty^{nm}$ is the stable value of the point $(n, m)$ of the spectral sequence (3) then for any $j \geq 0$,

$$H^j(G, V) = \bigoplus_{n+m=j} E_\infty^{nm}. \quad (5)$$

*1 Properties of the Lyndon-Hochschild-Serre spectral sequence*

First, we prove the properties of the spectral sequence (3) necessary for the proof of Theorem 1.

*Lemma 1. Let $p > h$ and $V$ be a simple $G$-module with highest weight $\lambda \in X_1(T)$. Then for all $m > 0$ there is an exact sequence of $G$ -modules*

$$0 \longrightarrow H^{m-1}(G^1, H^0(\lambda)/V)^{(-1)} \longrightarrow H^m(G^1, V)^{(-1)} \to H^m(G^1, H^0(\lambda))^{(-1)} \longrightarrow 0. \quad (6)$$

*Proof.* A simple $G$-module $V$ is the socle of the induced module $H^0(\lambda)$, i.e. there is a short exact sequence

$$0 \longrightarrow V \longrightarrow H^0(\lambda) \longrightarrow H^0(\lambda)/V \longrightarrow 0.$$

Consider the corresponding long cohomological exact sequence of $G^1$-cohomology

$$\cdots \longrightarrow H^m(G^1, V)^{(-1)} \longrightarrow H^m(G^1, H^0(\lambda))^{(-1)} \longrightarrow H^m(G^1, H^0(\lambda)/V)^{(-1)} \longrightarrow \cdots. \quad (7)$$

We apply induction on $m$. According to [18, 4.9], formula (6) is true for $m = 1$. If it is true for all $m \leq a-1$, then from the exactness of sequence (7) it follows that the sequence

$$0 \longrightarrow H^{a-1}(G^1, H^0(\lambda)/V)^{(-1)} \longrightarrow H^a(G^1, V)^{(-1)} \longrightarrow H^a(G^1, H^0(\lambda))^{(-1)} \longrightarrow \cdots \quad (8)$$

is also exact. If $H^a(G^1, H^0(\lambda))^{(-1)} = 0$ then as it can be seen from (8), that the sequence (6) is exact for $m = a$. If

$$H^a(G^1, H^0(\lambda))^{(-1)} \neq 0$$

then according to (1) and (2),

$$H^{a+1}(G^1, H^0(\lambda))^{(-1)} = 0.$$

In this case, from the long exact sequence (8) it follows that the sequence

$$0 \longrightarrow H^{a-1}(G^1, H^0(\lambda)/V)^{(-1)} \longrightarrow H^a(G^1, V)^{(-1)} \longrightarrow$$

$$\longrightarrow H^a(G^1, H^0(\lambda))^{(-1)} \longrightarrow H^a(G^1, H^0(\lambda)/V)^{(-1)} \longrightarrow 0$$

is exact.

We prove that $H^a(G^1, H^0(\lambda)/V)^{(-1)} = 0$. Indeed, the highest weight $\mu$ of any composition factor $H^0(\lambda)/V$ in Jantzen filtration is strongly linked to $\lambda$ and $\mu \preceq \lambda$ [19, p. 54]. Therefore,

$$H^a(G^1, H^0(\lambda)/V)^{(-1)} = 0.$$

Thus, the short sequence (6) is exact for any $m > 0$.

*Lemma 2. Let $p > h$ and $\lambda \in X_1(T)$. Then* $H^n(G, H^m(G^1, H^0(\lambda))^{(-1)}) = 0$ for all $n > 0$ и $m \geq 0$.

*Proof.* According to (1) and [12, Lemma 4.5], $G$-module $H^m(G^1, H^0(\lambda))^{(-1)}$ admits good filtration for all $m \geq 0$. It is well known that non-trivial induced modules are $G$-acyclic [14, Corol. 3.4], [15, Lemma 2.1, (iii)], [16, II.4.13, (1)]. Therefore,

$$H^n(G, H^m(G^1, H^0(\lambda))^{(-1)}) = 0$$

for all $n > 0$ and $m \geq 0$.

*Proposition 1. Let $p > h$ and $V$ be a non-trivial simple $G$ module with the highest weight from the restricted region. Then $E_2^{n,m} = 0$ for all $n > 0$ and for all $m \geq 0$.*

*Proof.* According to the formula (4), $E_2^{n,m} = H^n(G, H^m(G^1, V)^{(-1)})$. Let us prove the statements of the lemma by induction $m$. Since $V$ is a nontrivial simple $G^1$-module with highest restricted weight, then $H^0(G^1, V)^{(-1)} = 0$. Therefore, $E_2^{n,0} = 0$ for all $n > 0$. Assume that $E_2^{n,s} = 0$ for all $n > 0$ and $s < m$, we prove triviality $E_2^{n,m}$ for all $n > 0$. Consider a long cohomological sequence of $G$-cohomology that corresponds to a short exact sequence (6)

$$\cdots \longrightarrow H^n(G, H^{m-1}(G^1, H^0(\lambda)/V)^{(-1)})$$

$$\longrightarrow H^n(G, H^m(G^1, V)^{(-1)}) \longrightarrow H^n(G, H^m(G^1, H^0(\lambda))^{(-1)}) \longrightarrow \cdots. \quad (9)$$

According to Lemma 2, $H^n(G, H^m(G^1, H^0(\lambda))^{(-1)}) = 0$ for all $n > 0$. Then from the exactness of sequence (9) it follows that, for all $n > 0$, there is an isomorphism

$$H^n(G, H^m(G^1, V)^{(-1)}) \cong H^n(G, H^{m-1}(G^1, H^0(\lambda)/V)^{(-1)}). \tag{10}$$

Now we prove that $H^n(G, H^{m-1}(G^1, H^0(\lambda)/V)^{(-1)}) = 0$. By the induction hypothesis,

$$E_2^{n,s} = H^n(G, H^s(G^1, V)^{(-1)}) \cong H^n(G, H^{s-1}(G^1, H^0(\lambda)/V)^{(-1)}) = 0$$

for all $s < m$. This means that the socle of $G$-module $H^0(\lambda)/V$ has $G^1$-cohomology of all degrees up to the degree $m-2$, which direct summands have zero $G$-cohomology. Let $L(\mu) \subset soc_G H^0(\lambda)/V$. Then, applying Lemma 1 for $V = L(\mu)$, we obtain the following exact sequence:

$$0 \longrightarrow H^{m-2}(G^1, H^0(\mu)/L(\mu))^{(-1)} \longrightarrow$$
$$H^{m-1}(G^1, L(\mu))^{(-1)} \longrightarrow H^{m-1}(G^1, H^0(\mu))^{(-1)} \longrightarrow 0. \tag{11}$$

Let $H^{m-2}(G^1, H^0(\mu)/L(\mu))^{(-1)} = 0$, then

$$H^{m-1}(G^1, L(\mu))^{(-1)} \cong H^{m-1}(G^1, H^0(\mu))^{(-1)},$$

Hence $H^{m-1}(G^1, L(\mu))^{(-1)}$, as $G$-module, admits a good filtration. Thus,

$$H^n(G, H^{m-1}(G^1, L(\mu))^{(-1)}) = 0.$$

If $H^{m-2}(G^1, H^0(\mu)/L(\mu))^{(-1)} \neq 0$ then by induction hypothesis,

$$H^n(G, H^{m-2}(G^1, H^0(\mu)/L(\mu))^{(-1)}) = 0 \text{ for all } L(\mu) \subset soc_G H^0(\lambda)/V.$$

Then, due to the exactness of the sequence (11),

$$H^n(G, H^{m-1}(G^1, H^0(\lambda)/V)^{(-1)}) \cong \bigoplus_{L(\mu) \subset soc_G H^0(\lambda)/V} H^n(G, H^{m-1}(G^1, H^0(\mu))^{(-1)}) = 0.$$

Since for all $L(\mu) \subset soc_G H^0(\lambda)/V$, $G$-module $H^{m-1}(G^1, H^0(\mu))^{(-1)}$ admits a good filtration, then $H^n(G, H^{m-1}(G^1, H^0(\lambda)/V)^{(-1)}) = 0$. Therefore, according to the formula (10), $H^n(G, H^m(G^1, V)^{(-1)}) = 0$. Thus, it is proved that $E_2^{n,m} = H^n(G, H^m(G^1, H^0(\lambda)/V)^{(-1)}) = 0$ for all $n > 0$ and for all $m \geq 0$.

*Lemma 3. Let $p > h$ and $V$ be a simple $G$-module with highest weight from the restricted region. Then $E_2^{0,j} = E_\infty^{0,j}$ and $H^j(G, V) = E_2^{0,j}$ for all $j \geq 0$.*

*Proof.* According to the definition, $E_{i+1}^{n,m}$ is the cohomology of the sequence

$$E_i^{n-i,m+i-1} \to E_i^{n,m} \to E_i^{n+i,m-i+1}.$$

Then it is obvious that $E_{j+2}^{0,j} = E_\infty^{0,j}$. Thus, $E_2^{0,j} = E_\infty^{0,j}$, if

$$E_2^{0,j} = E_3^{0,j} = \cdots = E_{j+2}^{0,j}. \tag{12}$$

Let us prove the condition (12) by induction on $j$. For $j = 0$, this is obvious. Let (12) holds for all $j < a$. Let us prove that it is true for $j = a$. Since, according the induction hypothesis, $E_{a+2}^{n,m}$ is the cohomology of sequence

$$E_2^{n-a-1,m+a} \to E_2^{n,m} \to E_2^{n+a+1,m-a},$$

then $E_{a+1}^{0,a} = E_{a+2}^{0,a}$, if

$$E_2^{-a-1,2a} = E_2^{a+1,0} = 0 \text{ whenever } E_2^{0,a} \neq 0.$$

Let $E_2^{0,a} \neq 0$. Thus, it is obvious that $E_2^{-a-1,2a} = 0$ and, according to the Proposition 1, $E_2^{a+1,0} = 0$. Therefore, the condition (12) is true for all non-negative $j$, and $E_2^{0,j} = E_\infty^{0,j}$ for all $j \geq 0$.

According to the Proposition 1, $E_2^{n,m} = 0$ for all $n > 0$ and $m \geq 0$. If $j = n + m$, then $E_2^{j-m,m} = 0$ for $0 \leq m < j-1$. Then $E_\infty^{j-m,m} = E_2^{j-m,m} = 0$ for all $0 \leq m \leq j-1$. Thus, according to the formula (5), $H^j(G, V) = E_\infty^{0,j} = E_2^{0,j}$ for all $j \geq 0$.

*2 Proof of Theorem 1*

According to Lemma 3, $H^j(G,V) = E_2^{0,j}$ for all $j \geq 0$. Using the formula (4), we obtain

$$E_2^{0,j} \cong H^0(G, H^j(G^1, V)^{(-1)}).$$

Since

$$H^0(G, H^j(G^1, V)^{(-1)}) \cong Hom_G(k, H^j(G^1, V)^{(-1)})$$

then $E_2^{0,j} \cong Hom_G(k, H^j(G^1, V)^{(-1)})$. Therefore,

$$H^j(G,V) \cong Hom_G(k, H^j(G^1, V)^{(-1)}).$$

The proof of Theorem 1 is complete.

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## Алгебралық группалар үшін жәй модульдердің когомологиялары

Мақалада жәй бірбайланысқан алгебралық группалардың коэффициенттерінің жәй модульдердегі когомологияларын зерттеуге байланысты сұрақтар қарастырылған. Оларды есептеудің әртүрлі әдістері бар. Тиімді есептеу әдістердің бірі – бұл, инфинитезимальді ішкі группаға – берілген алгебралық группаның Фробениус ядросына қатысты Линдон–Хохшильд–Серр спектрлік тізбектерінің қасиеттерін және әртүрлі когомологиялық дәл тізбектердің қасиеттерін пайдалану. Авторлар үлкен салмағы шектелген жәй модульдерге қатысты алгебралық группа модульдерінің әртүрлі қысқа дәл және сәйкесті ұзын дәл когомологиялық тізбектердің қасиеттерін зерттеді. Фробениус ядросының коэффициенттері үлкен салмағы шектелген жәй модульдердегі когомологияларының кейбір қасиеттері сипатталды. Сонымен қатар, үлкен салмағы шектелген жәй модульдер үшін Линдон–Хохшильд–Серр спектрлік тізбектерінің бірінші квадранттағы қасиеттері зерттелді. Спектрлік тізбектің бірінші квадранттағы нүктелерінің шектік мәндері есептелді. Сипаттамасы $p > h$ алгебралық тұйық $k$ өрісіне қатысты жәй бірбайланысқан, түбірлер жүйесі $R$ келтірілмеген $G$ алгебралық группасы және жоғары салмағы шектелген жәй $V$ $G$-модулі үшін:

$$H^j(G,V) \cong Hom_G(k, H^j(G^1,V)^{(-1)}) \text{ барлығы } j \geq 0,$$

мұндағы $G^1$ – $G$ үшін Фробениус бейнелеуінің ядросы, $h$ – $R$ түбірлер жүйесінің Кокстер саны. Бұл изоморфизм $G$ группасының коэффициенттері жоғары салмағы шектелген жәй модульдердегі когомологияларын есептеуді $G^1$ Фробениус ядросының сәйкесті когомологияларын есептеуге әкеледі.

*Кілт сөздер:* алгебралық группа, Шевалле группасы, Ли группасының көрінісі, Фробениус ядросы, жәй модуль, когомология, спектрлік тізбек, дәл тізбек, шектелген салмақ.

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## Когомологии простых модулей для алгебраических групп

В статье рассмотрены вопросы, касающиеся изучения когомологии простых односвязных алгебраических групп с коэффициентами в простых модулях. Существуют различные методы их вычисления. Одним из эффективных методов является использование свойств спектральной последовательности Линдона–Хохшильда–Серра относительно инфинитезимальной подгруппы – ядра Фробениуса данной алгебраической группы и свойств различных точных когомологических последовательностей. Авторами изучены свойства различных коротких точных и соответствующих длинных точных когомологических последовательностей модулей над алгебраической группой, связанных с простыми модулями со старшими ограниченными весами. Описаны некоторые свойства когомологии ядра Фробениуса с коэффициентами в простых модулях со старшими ограниченными весами. Кроме того, исследованы свойства спектральной последовательности Линдона–Хохшильда–Серра на первом квадранте для простых модулей со старшими ограниченными весами. Описаны предельные значения точек первого квадранта спектральной последовательности. Доказано, что для простой односвязной алгебраической группы $G$ над алгебраически замкнутым полем $k$ характеристики $p > h$ с неприводимой системой корней $R$ и для простого $G$-модуля $V$ со старшим ограниченным весом имеет место изоморфизм $G$-модулей

$$H^j(G,V) \cong Hom_G(k, H^j(G^1,V)^{(-1)}) \text{ для всех } j \geq 0,$$

где $G^1$ – ядро отображения Фробениуса для $G$; $h$ – число Кокстера системы $R$. Данный изоморфизм позволяет свести вычисление когомологии группы $G$ с коэффициентами в простых модулях с ограниченными старшими весами к вычислению соответствующих когомологии ядра Фробениуса $G^1$.

*Ключевые слова:* алгебраическая группа, группа Шевалле, представление группы Ли, ядро Фробениуса, простой модуль, когомология, спектральная последовательность, точная последовательность, ограниченный вес.

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# Deformations of the three-dimensional Lie algebra $\mathfrak{sl}(2)$

Deformation is one of key questions of the structural theory of algebras over a field. Especially, it plays a important role in the classification of such algebras. In odd characteristics of algebraically closed fields, local deformations of classical Lie algebras are completely described. Local deformations are also known for classical Lie algebras with a homogeneous root system over an algebraically closed field of characteristic 2, except for the three-dimensional Lie algebra $\mathfrak{sl}(2)$. In the characteristic 2, deformations of Lie algebras with an non-homogeneous root system are calculated only for Lie algebras of small ranks. In this paper we investigate deformations of the three-dimensional classical Lie algebra $\mathfrak{sl}(2)$ over an algebraically closed field $k$ of characteristic $p = 2$. We also describe three-dimensional two-sided Alia algebras associated with Lie algebra $\mathfrak{sl}(2)$ in the characteristics 2 and 3. It is proved that, in characteristic 2, the space of local deformations of the Lie algebra $\mathfrak{sl}(2)$ is five-dimensional. The structural specialty of the second cohomology space of the adjoint representation of the Lie algebra $\mathfrak{sl}(2)$ are analyzed. In particular, the subspace of cosets of restricted cocycles is described. It is proved that the subspace of classes of restricted cocycles is two-dimensional and the corresponding local deformations are restricted Lie algebras in the sense of Jacobson. It was found that a family of simple three-dimensional unrestricted Lie algebras correspond to unrestricted non-trivial cocycles. In characteristics 2 and 3, three-dimensional two-sided Alia algebras that are non-isomorphic to the Lie algebra $\mathfrak{sl}(2)$ are constructed. In the process of the study, a complete description of the space of all derivations of the Lie algebra $\mathfrak{sl}(2)$ is obtained.

*Keywords:* Lie algebra, module, representation, derivation, outer derivation, deformation, restricted deformation, cohomology, cocycle, commutative cocycle, Alia algebra.

*Introduction*

Over an algebraically closed field of characteristic zero, classical Lie algebras are rigid. Deformations of classical Lie algebras in positive characteristics were studied in [1–10]. In [11–14] deformations of Cartan type Lie algebras are studied.

In this paper deformations of the three-dimensional classical Lie algebra $\mathfrak{g} = \mathfrak{sl}(2)$ over an algebraically closed field $k$ of characteristic $p = 2$ are calculated. It is well-known that, in the case when $p > 2$, the Lie algebra $\mathfrak{sl}(2)$ is rigid [6]. We prove that in characteristic 2 the Lie algebra $\mathfrak{sl}(2)$ admits a five-dimensional space of local deformations (Theorem 1). To prove Theorem 1, we use information on the structure of the space of outer derivations of the Lie algebra $\mathfrak{g}$. In section 1 we give a complete description of the space of outer differentiations of the Lie algebra $\mathfrak{g}$ (Proposition 1). The dimension of the space of outer derivations of the Lie algebra $\mathfrak{g}$ was previously calculated in [15]. In section 2 the spaces of usual and restricted second cohomologies of the Lie algebra $\mathfrak{g}$ with coefficients in the adjoint representation are calculated. According to the general theory of deformation, a necessary condition for the deformation of a Lie algebra is the non-triviality of its second cohomology with coefficients in the adjoint representation. However, in the general case, the correspondence between the 2-cocycle classes of the second cohomology for the adjoint representation and the deformations of the Lie algebra is not one-to-one [8]. In this connection, we prove that the parametrizability of local deformations of a restricted Lie algebra $\mathfrak{g}$ by elements of the second cohomology $H^2(\mathfrak{g}, \mathfrak{g})$ (Lemma 1). By restricted local deformations of a Lie algebra $\mathfrak{g}$ we mean deformations corresponding to elements of the second restricted cohomology $H^2_*(\mathfrak{g}, \mathfrak{g})$. A restricted cohomology of a restricted Lie algebra was first introduced by Hochschild in [16]. In the last section 3 the space of commutative cocycles with coefficients in $k$ is calculated (Proposition 2). Commutative cocycles play an important role in the structural theory of two-sided Alia algebras [17–19] and in the second cohomology groups of current Lie algebras [20]. An algebra $(A, \circ)$ is called a two-sided Alia algebra if the identities

$$[a, b] \circ c + [b, c] \circ a + [c, a] \circ b = 0,$$

$$a \circ [b, c] + b \circ [c, a] + c \circ [a, b] = 0$$

hold, where $[a, b] = a \circ b - b \circ a$ is the usual commutator. The two-sided Alia algebra is Lie-admissible, i.e. the space $A$ becomes a Lie algebra respect to the multiplication $[a, b] = a \circ b - b \circ a$. The Lie algebra deformed by commutative cocycles is a two-sided Alia algebra non-isomorphic to the Lie algebra itself. It was proved in [17] that if $p \neq 2, 3$, then among the classical Lie algebras only the three-dimensional Lie algebra $\mathfrak{sl}(2)$ admits commutative cocycles. Commutative cocycles for some important classes of Lie algebras such as current Lie algebras, Kac–Moody algebras, finite-dimensional semi-simple algebras were studied in [21].

Let $\mathfrak{g}$ be a Lie algebra over an algebraically closed field $k$ of characteristic $p$. The space of $i$-dimensional cochains $C^i(\mathfrak{g}, \mathfrak{g})$ of an ordinary cochain complex is defined as the space of skew-symmetric poly-linear functions $\psi : \bigwedge^i(\mathfrak{g}) \to \mathfrak{g}$ with differential $d$ defined by

$$d\psi(l_1, l_2, \cdots, l_{i+1}) = \sum_{j=1}^{i+1} (-1)^j [l_j, \psi(l_1, \cdots, \widehat{l_j}, \cdots, l_{i+1})] +$$

$$\sum_{p<q} (-1)^{p+q} \psi([l_p, l_q], \cdots, \widehat{l_p}, \cdots, \widehat{l_q}, \cdots, l_{i+1}),$$

where $l_1, l_2, \cdots, l_{i+1} \in \mathfrak{g}$, and the notation $\widehat{l_j}$ means that the element $l_j$ should be omitted.

Let

$Z^i(\mathfrak{g}, \mathfrak{g}) = Ker\, d \cap C^i(\mathfrak{g}, \mathfrak{g})$ is the space of $i$-dimensional cocycles,

$B^i(\mathfrak{g}, \mathfrak{g}) = Im\, d \cap C^i(\mathfrak{g}, \mathfrak{g})$ is the $i$-dimensional cochains,

and $H^i(\mathfrak{g}, \mathfrak{g}) = Z^i(\mathfrak{g}, \mathfrak{g})/B^i(\mathfrak{g}, \mathfrak{g})$ is the $i$-dimensional cohomologies.

If $\omega \in Z^1(\mathfrak{g}, \mathfrak{g})$, $\psi \in Z^2(\mathfrak{g}, \mathfrak{g})$ then

$$-\omega([l_1, l_2]) - [l_1, \omega(l_2)] + [l_2, \omega(l_1)] = 0 \text{ for all } l_1, l_2 \in \mathfrak{g}, \tag{1}$$

$$-\psi([l_1, l_2], l_3) + \psi([l_1, l_3], l_2) - \psi([l_2, l_3], l_1) - \tag{2}$$

$$[l_1, \psi(l_2, l_3)] + [l_2, \psi(l_1, l_3)] - [l_3, \psi(l_1, l_2)] = 0 \text{ for all } l_1, l_2, l_3 \in \mathfrak{g}.$$

Let now $\mathfrak{g} = \mathfrak{sl}(2)$. Choose a basis $\{e, h, f\}$ of the Lie algebra $\mathfrak{g}$ with the multiplication table $[h, e] = 2e$, $[h, f] = -2f$, $[e, f] = h$. In the dual space $\mathfrak{g}^*$ we choose the dual basis $\{e^*, h^*, f^*\}$ for the basis $\{e, h, f\}$. We identify the space $C^i(\mathfrak{g}, \mathfrak{g})$ with the space $\bigwedge^i(\mathfrak{g}^*) \otimes \mathfrak{g}$. The cohomological class of the cocycle $\psi \in Z^i(\mathfrak{g}, \mathfrak{g})$ is denoted by $[\psi]$.

## *1 Derivations*

*Proposition 1. Let $\mathfrak{g} = \mathfrak{sl}_2(k)$ be the three-dimensional classical Lie algebra over an algebraically closed field $k$ of characteristic $p = 2$. Then the following isomorphisms of the vector spaces over $k$ hold:*

*(a)* $Z^1(\mathfrak{g}, \mathfrak{g}) \cong \langle \omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6 \rangle_k$;

*(b)* $H^1(\mathfrak{g}, \mathfrak{g}) \cong \langle [\omega_1], [\omega_2], [\omega_3], [\omega_4] \rangle_k$,

*where* $\omega_1 = e^* \otimes e + h^* \otimes h$, $\omega_2 = e^* \otimes f$, $\omega_3 = f^* \otimes e$, $\omega_4 = f^* \otimes f + h^* \otimes h$, $\omega_5 = e^* \otimes h$, $\omega_6 = f^* \otimes h$.

*Proof.* (*a*) First we prove that the cochains $\omega_1, \omega_2, \cdots, \omega_6$ are cocycles. To do this, it is sufficient to check condition (1) for the basis elements $e, h, f$. Indeed, since

$$-\omega_1([e, h]) - [e, \omega_1(h)] + [h, \omega_1(e)] = -2\omega_1(e) - [e, h] + [h, e] = -2e - 2e + 2e = 0,$$

$$-\omega_1([e, f]) - [e, \omega_1(f)] + [f, \omega_1(e)] = -\omega_1(h) + [f, e] = -h - h = 0,$$

$$-\omega_1([h, f]) - [h, \omega_1(f)] + [f, \omega_1(h)] = 2\omega_1(f) + [f, h] = 2f = 0,$$

then $\omega_1$ is a cocycle. Similarly, the condition (1) is easily verified for other cochains.

Let

$$\omega = x_1 e^* \otimes e + x_2 e^* \otimes h + x_3 e^* \otimes f + y_1 h^* \otimes e + y_2 h^* \otimes h + y_3 h^* \otimes f$$

$$+ z_1 f^* \otimes e + z_2 f^* \otimes h + z_3 f^* \otimes f \in Z^1(\mathfrak{g}, \mathfrak{g}),$$

where $x_j, y_j, z_j \in k$. The following implications hold:

$$-\omega([e, h]) - [e, \omega(h)] + [h, \omega(e)] = 0 \Longrightarrow y_3 = 0,$$

$$-\omega([e,f]) - [e,\omega(f)] + [f,\omega(e)] = 0 \Longrightarrow y_1 = y_3 = 0,\ x_1 + y_2 + z_3 = 0,$$
$$-\omega([h,f]) - [h,\omega(f)] + [f,\omega(h)] = 0 \Longrightarrow y_1 = 0.$$

Therefore, from the condition (1) it follows that $y_1 = y_3 = 0$ and $x_1 + y_2 + z_3 = 0$. These equalities form a linear system with respect to $x_j, y_j, z_j \in k$. The rank of this system is 3, so it has a six-dimensional space of solutions. As a basis of $Z^1(\mathfrak{g},\mathfrak{g})$, one can choose cocycles $\omega_1, \omega_2, \cdots, \omega_6$.

($b$) Let $\omega = \sum_{j=1}^{6} a_j\omega_j \in Z^1(\mathfrak{g},\mathfrak{g})$, where $a_j \in k$. Suppose that $\omega \in B^1(\mathfrak{g},\mathfrak{g})$. Then for the basis elements of Li algebra the equalities

$$\omega(e) = [e, b_1e + b_2h + b_3h],\ \omega(h) = [h, b_1e + b_2h + b_3h],\ \omega(f) = [f, b_1e + b_2h + b_3h]$$

hold, where $b_j \in k$. From these equalities it follows that $a_1 = a_2 = a_3 = a_4 = 0$, $a_5 = c_1$, $a_6 = c_3$. Thus, the cocycles $\omega_1, \omega_2, \omega_3, \omega_4$ are linearly independent and its cosets form the basis of the space $H^1(\mathfrak{g},\mathfrak{g})$.

The proof of Proposition 1 is complete.

## *2 Local deformations*

*Theorem 1. Let $\mathfrak{g} = \mathfrak{sl}(2)$ be the three-dimensional classical Lie algebra over an algebraically closed field $k$ of characteristic $p = 2$. Then the following isomorphisms of vector spaces over $k$ hold:*

($a$) $Z^2(\mathfrak{g},\mathfrak{g}) \cong \langle\psi_1, \psi_2, \psi_3, \psi_4, \psi_5, \psi_6, \psi_7, \psi_8\rangle_k$;

($b$) $H^2(\mathfrak{g},\mathfrak{g}) \cong \langle[\psi_1], [\psi_2], [\psi_3], [\psi_4], [\psi_5]\rangle_k$;

($c$) $H^2_*(\mathfrak{g},\mathfrak{g}) \cong \langle[\psi_5], [\psi_6]\rangle_k$,

*where* $\psi_1 = e^* \wedge h^* \otimes e + h^* \wedge f^* \otimes f$, $\psi_2 = e^* \wedge h^* \otimes f$, $\psi_3 = h^* \wedge f^* \otimes e$, $\psi_4 = e^* \wedge f^* \otimes f$, $\psi_5 = e^* \wedge f^* \otimes e$, $\psi_6 = e^* \wedge h^* \otimes h$, $\psi_7 = h^* \wedge f^* \otimes h$, $\psi_8 = e^* \wedge f^* \otimes h$.

*Proof.* ($a$) Since for any cochain $\psi \in C^2(\mathfrak{g},\mathfrak{g})$,

$$-\psi([e,h],f) + \psi([e,f],h) - \psi([h,f],e) = 0$$

and

$$-[e,\psi_1(h,f)] + [h,\psi_1(e,f)] - [f,\psi_1(e,h)] = [e,f] - [f,e] = 0,$$
$$-[e,\psi_2(h,f)] + [h,\psi_2(e,f)] - [f,\psi_2(e,h)] = -[f,f] = 0,$$
$$-[e,\psi_3(h,f)] + [h,\psi_3(e,f)] - [f,\psi_3(e,h)] = -[e,e] = 0,$$
$$-[e,\psi_4(h,f)] + [h,\psi_4(e,f)] - [f,\psi_4(e,h)] = [h,f] = 0,$$
$$-[e,\psi_5(h,f)] + [h,\psi_5(e,f)] - [f,\psi_5(e,h)] = [h,e] = 0,$$
$$-[e,\psi_6(h,f)] + [h,\psi_6(e,f)] - [f,\psi_6(e,h)] = -[f,h] = 0,$$
$$-[e,\psi_7(h,f)] + [h,\psi_7(e,f)] - [f,\psi_7(e,h)] = -[e,h] = 0,$$
$$-[e,\psi_8(h,f)] + [h,\psi_8(e,f)] - [f,\psi_8(e,h)] = [h,h] = 0,$$

then by (2), the cochains $\psi_1, \psi_2, \cdots, \psi_8$ are cocycles.

Let

$$\psi = x_1e^* \wedge h^* \otimes e + x_2e^* \wedge h^* \otimes h + x_3e^* \wedge h^* \otimes f + y_1e^* \wedge f^* \otimes e + y_2e^* \wedge f^* \otimes h + y_3e^* \wedge f^* \otimes f$$
$$+z_1h^* \wedge f^* \otimes e + z_2h^* \wedge f^* \otimes h + z_3h^* \wedge f^* \otimes f \in Z^2(\mathfrak{g},\mathfrak{g}),$$

where $x_j,\ y_j,\ z_j \in k$. Then from the cocycle condition (2) it follows that $x_1 + z_3 = 0$. Therefore, $\dim Z^2(\mathfrak{g},\mathfrak{g}) = 9 - 1 = 8$. The cocycles $\psi_j$, $j = 1, 2, \cdots, 8$ form the basis of $Z^2(\mathfrak{g},\mathfrak{g})$.

($b$) Let

$$\psi = \sum_{j=1}^{8} a_j\psi_j \in Z^2(\mathfrak{g},\mathfrak{g})$$

and

$$\omega = b_1e^* \otimes e + b_2e^* \otimes h + b_3e^* \otimes f + b_4h^* \otimes e + b_5h^* \otimes h + b_6h^* \otimes f$$
$$+b_7f^* \otimes e + b_8f^* \otimes h + b_9f^* \otimes f \in C^1(\mathfrak{g},\mathfrak{g}),$$

where $a_j, b_j \in k$. Then from the condition $\psi = d\omega$ it follows that $a_1 = a_2 = a_3 = 0$, $a_4 = b_6$, $a_5 = b_4$, $a_6 = b_6$, $a_7 = b_4$, $a_8 = b_1 + b_5 + b_9$. Therefore the cocycles

$$\psi_1, \psi_2, \psi_3, \psi_4, \psi_5$$

are linear independent and its cosets form a basis of $H^2(\mathfrak{g}, \mathfrak{g})$.

($c$) For the adjoint $\mathfrak{g}$-module $\mathfrak{g}$ there is the following Hochschild exact sequence [16]:

$$0 \to H^1_*(\mathfrak{g}, \mathfrak{g}) \to H^1(\mathfrak{g}, \mathfrak{g}) \xrightarrow{T} S(\mathfrak{g}, \mathfrak{g}^{\mathfrak{g}}) \to H^2_*(\mathfrak{g}, \mathfrak{g}) \to H^2(\mathfrak{g}, \mathfrak{g}) \xrightarrow{D} S(\mathfrak{g}, H^1(\mathfrak{g}, \mathfrak{g})),$$

where

$$S(\mathfrak{g}, V) = \{u \,:\, \mathfrak{g} \to V \mid u(\alpha_1 l_1 + \alpha_2 l_2) = \alpha_1^p u(l_1) + \alpha_2^p u(l_2),\, \alpha_1, \alpha_2 \in k,\, l_1, l_2 \in \mathfrak{g}\}$$

is a space of semi-linear maps from $\mathfrak{g}$ to $V$. The maps $T$ and $D$ defined by

$$T_\omega(l_1) = [l_1^{p-1}, \omega(l_1)] - \omega(l_1^{[p]}),$$

$$D_\psi(l_1)l_2 = \sum_{j=0}^{p-1} (ad\, l_1)^j (\psi(l_1, (ad\, l_1)^{p-1-j}(l_2))) - \psi(l_1^{[p]}, l_2),$$

where $l_1, l_2 \in \mathfrak{g}$, $\omega \in Z^1(\mathfrak{g}, \mathfrak{g})$, and $\psi \in Z^2(\mathfrak{g}, \mathfrak{g})$. In particular, if $p = 2$ then

$$T_\omega(l_1) = [l_1, \omega(l_1)] - \omega(l_1^{[2]}), \tag{3}$$

$$D_\psi(l_1)l_2 = \psi(l_1, [l_1, l_2]) + [l_1, \psi(l_1, l_2)] - \psi(l_1^{[2]}, l_2). \tag{4}$$

If $D_\psi(l_1)$ is a inner derivation for the Lie algebra $\mathfrak{g}$ for some $l_1 \in \mathfrak{g}$ then It is obvious that the image of cocycle coset $[\psi]$ under the map

$$D \,:\, H^2(\mathfrak{g}, \mathfrak{g}) \to S(\mathfrak{g}, H^1(\mathfrak{g}, \mathfrak{g}))$$

is trivial.

By (3) and Proposition 1,

$$T_{\omega_1} = T_{\omega_4} = u,\, T_{\omega_2} = \omega_5,\, T_{\omega_3} = \omega_6,$$

where $u$ is a semi-linear map defined by $u(h) = h$, $u(e) = u(f) = 0$. Therefore, $\Im T \cong S(\mathfrak{g}, \mathfrak{g}^{\mathfrak{g}})$. Then from the previous exact sequence it follows that the following sequence is exact:

$$0 \to H^2_*(\mathfrak{g}, \mathfrak{g}) \to H^2(\mathfrak{g}, \mathfrak{g}) \xrightarrow{D} S(\mathfrak{g}, H^1(\mathfrak{g}, \mathfrak{g})). \tag{5}$$

Using (4) and the statement ($a$) of Proposition 1, we get

$$D_{\psi_1}(h) = \omega_1 + \omega_4,\, D_{\psi_2}(h) = \omega_2,\, D_{\psi_3}(h) = \omega_3,\, D_{\psi_4}(h) = \omega_5,\, D_{\psi_5}(h) = \omega_6.$$

Then, by the statements ($a$) and ($b$) of Proposition 1, the maps $D_{\psi_4}(h)$, $D_{\psi_5}(h)$ are inner derivations of $\mathfrak{g}$, and the maps $D_{\psi_1}(h)$, $D_{\psi_2}(h)$, $D_{\psi_3}(h)$ are outer derivations. Therefore, $[\psi_4], [\psi_5] \in H^2_*(\mathfrak{g}, \mathfrak{g})$ and $[\psi_1], [\psi_2]$, $[\psi_3] \notin H^2_*(\mathfrak{g}, \mathfrak{g})$. Then the statement ($c$) follows from the exact sequence (5).

The proof of Theorem 1 is complete.

*Lemma 1. The cocycle cosets of the space $H^2(\mathfrak{g}, \mathfrak{g})$ define nontrivial local deformations of the Lie algebra $\mathfrak{g}$.*

*Proof.* By Theorem 1,

$$H^2(\mathfrak{g}, \mathfrak{g}) \cong \langle [\psi_1],\, [\psi_2],\, [\psi_3],\, [\psi_4],\, [\psi_5] \rangle_k.$$

Denote by $\mathfrak{sl}(2, t_j)$ the local deformation of the Lie algebra $\mathfrak{g}$ corresponding to the cocycle coset $[\psi_j]$. For the basis elements of the Lie algebra $\mathfrak{sl}(2, t_j)$, we also use the notation of the basis elements of the Lie algebra $\mathfrak{g}$. The multiplication of the Lie algebra $\mathfrak{sl}(2, t_j)$ is defined by

$$[l_1, l_2]_{t_j} = [l_1, l_2] + t_j \psi_j(l_1, l_2),\;\; l_1, l_2 \in \mathfrak{sl}(2, t_j),\, t_j \in k.$$

First, we consider the Lie algebra $\mathfrak{sl}(2, t_1)$. It is a simple Lie algebra. Therefore, it corresponds to a nontrivial local deformation of the Lie $\mathfrak{g}$.

If $j \in \{2,3,4,5\}$ then $\mathfrak{sl}(2,t_j)$ is not a simple Lie algebra. Any homomorphism $\mathfrak{sl}(2,t_j) \longrightarrow \mathfrak{g}$ is not one-to-one. Indeed, if $\phi : \mathfrak{sl}(2,t_j) \longrightarrow \mathfrak{g}$ is a homomorphism of Lie algebras, then $\phi(h) = 0$ if $j = 2,3$, $t_4\phi(h)+\phi(f) = 0$ if $j = 4$, and $t_5\phi(h)+\phi(e) = 0$ if $j = 5$. Hence, the cocycle cosets $\psi_2$, $\psi_3$, $\psi_4$, $\psi_5$ define nontrivial local deformations of the Lie algebra $\mathfrak{g}$.

The proof of Lemma 1 is complete.

*Remark 1.* By Theorem 1, $H^2_*(\mathfrak{g},\mathfrak{g})$ is a proper subspace of $H^2(\mathfrak{g},\mathfrak{g})$. This means that not all local deformations of the Lie algebra $\mathfrak{g}$ admit a restricted structure.

*3 Deformations by commutative cocycles*

Let $\mathfrak{g}$ be a Lie algebra over an algebraically closed field $k$ of characteristic $p$ and $V$ be a $\mathfrak{g}$-module. A bilinear map $\eta : \mathfrak{g} \times \mathfrak{g} \longrightarrow V$ satisfying conditions

$$\eta([l_1,l_2],l_3) + \eta([l_2,l_3],l_1) + \eta([l_3,l_1],l_2) = 0, \tag{6}$$

$$\eta(l_1,l_2) = \eta(l_2,l_1), \tag{7}$$

where $l_1,\ l_2,\ l_3 \in \mathfrak{g}$, is called *a commutative cocycle with coefficients in* $V$. Let $Z^2_{com}(\mathfrak{g},M)$ be the space of the commutative cocycles with coefficients in $V$ and $Z^2_{com}(\mathfrak{g}) = Z^2_{com}(\mathfrak{g},k)$. If $p \neq 2,3$, then among classical Lie algebras only a three-dimensional classical Lie algebra $\mathfrak{sl}(2)$ admits commutative cocycles and $\dim Z^2_{com}(\mathfrak{sl}(2)) = 5$ [17, Theorem 1]. In characteristic $p = 2$, from the condition (6) it follows that if $\eta \in Z^2_{com}(\mathfrak{sl}(2))$ then $\eta(h,h) = 0$. Hence, using the condition (7), we get the following

*Proposition 2. Let $\mathfrak{g} = \mathfrak{sl}(2)$ be the three-dimensional classical Lie algebra over an algebraically closed field $k$ of characteristic $p = 2$. Then*

$$Z^2_{com}(\mathfrak{g}) \cong \langle \eta_i : i = 1,\cdots,5\rangle_k,$$

*where*

$$\eta_1(e,e) = 1;\ \eta_2(e,h) = \eta_2(h,e) = 1;\ \eta_3(e,f) = \eta_3(f,e) = 1;$$

$$\eta_4(h,f) = \eta_4(f,h) = 1;\ \eta_5(f,f) = 1$$

*(not specified components are equal to zero).*

*Remark 2.* A statement similar to Proposition is also true in the case of characteristic $p = 3$. Indeed, according to (6), any commutative cocycle $\eta$ satisfies the equality $\eta(e,f) = \eta(h,h)$. Therefore, a basic commutative cocycle $\eta_3$ may be chosen so that the equalities

$$\eta_3(e,f) = \eta_3(f,e) = \eta_3(h,h) = 1$$

hold.

*Remark 3.* In the space $\mathfrak{g} = \mathfrak{sl}(2)$ the commutative cocycles of $Z^2_{com}(\mathfrak{g},\mathfrak{g})$ define Lie-admissible two sided Alia algebras non-isomorphic to $\mathfrak{g}$ [17]. Since

$$Z^2_{com}(\mathfrak{g},\mathfrak{g}) \cong Z^2_{com}(\mathfrak{g}) \otimes \mathfrak{g}$$

then, according to Proposition 2 and Remark 2, in characteristics $p = 2,3$ there exist three-dimensional Lie-admissible Alia algebras non-isomorphic to the Lie algebra $\mathfrak{sl}(2)$.

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## Үш өлшемді $\mathfrak{sl}(2)$ Ли алгебрасының деформациялары

Деформация өріске қатысты алгебралардың құрылымдық теориясының маңызды сұрақтарының бірі болып табылады. Әсіресе, осындай алгебралардың классификациясындағы оның орны ерекше. Сипаттамасы тақ алгебралық тұйық өрістерде классикалық Ли алгебраларының локальді деформациялары толық есептелген. Сол сияқты, сипаттамасы 2-ге тең өрісте түбірлер жүйесі біртекті классикалық Ли алгебраларының локальді деформациялары да $\mathfrak{sl}(2)$ Ли алгебрасынан басқа жағдайларда белгілі. Сиаттамасы 2-ге тең өрісте түбірлер жүйесі біртекті емес классикалық Ли алгебраларының локальді деформациялары тек рангі төмен Ли алгебраларында есептелген. Мақалада сипаттамасы $p = 2$ алгебралық тұйық $k$ өрісіндегі үш өлшемді классикалық $\mathfrak{sl}(2)$ Ли алгебрасының деформациялары зерттелді. 2 және 3-ке тең өріс сипаттамаларында, $\mathfrak{sl}(2)$ Ли алгебрасымен байланысты үш өлшемді екі жақты Алиа алгебралары қарастырылған. Сипаттамасы 2-ге тең өрісте $\mathfrak{sl}(2)$ Ли алгебрасының локальді деформациялары кеңістігінің бес өлшемді екені дәлелденді. $\mathfrak{sl}(2)$ Ли алгебрасының кірiктiрiлген модулiнiң екiншi когомологиялар кеңiстiгiнiң құрылымдық ерекшеліктері талданды. Дербес жағдайда, шектелген коциклдер кластарының ішкі кеңістігі сипатталды. Шектелген коциклдер кластарының ішкі кеңістігінің екі өлшемді және сәйкесті локальді деформациялардың Джекобсон мағынасында шектелген Ли алгебралары екені дәлелденді. Шектелмеген тривиаль емес коциклдерге жәй үш өлшемді Ли алгебраларының үйірі сәйкес келетіні анықталды. 2 және 3-ке тең сипаттамаларда, $\mathfrak{sl}(2)$ Ли алгебрасына изоморфты емес үш өлшемді екі жақты Алиа алгебралары құрылды. Жүргізілген зерттеулер нәтижесінде $\mathfrak{sl}(2)$ Ли алгебрасының барлық дифференциалдаулар кеңістігінің толық сипаттамасы алынды.

*Кілт сөздер:* Ли алгебрасы, модуль, көрініс, дифференциалдау, сыртқы дифференциалдау, деформация, шектелген деформация, когомология, коцикл, коммутативті коцикл, Алиа алгебрасы.

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# Деформации трехмерой алгебры Ли $\mathfrak{sl}(2)$

Деформация является одним из ключевых вопросов структурной теории алгебр над полем. Особенно важную роль она играет при классификации таких алгебр. В нечетных характеристиках алгебраически замкнутых полей локальные деформации классических алебр Ли описаны полностью. Также известны локальные деформации классических алгебр Ли с однородной системой корней над алгебраически замкнутым полем характеристики 2, кроме трехмерной алгебры Ли $\mathfrak{sl}(2)$. В характеристике 2 деформации алгебр Ли с неоднородной системой корней вычислены только для алгебр Ли малых рангов. В статье изучены деформации трехмерной классической алгебры Ли $\mathfrak{sl}(2)$ над алгебраически замкнутым полем $k$ характеристики $p = 2$. Описаны трехмерные двусторонние алгебры Алиа, связанные с алгеброй Ли $\mathfrak{sl}(2)$ в характеристиках 2 и 3. Доказано, что в характеристике 2 пространство локальных деформаций алгебры Ли $\mathfrak{sl}(2)$ пятимерно. Проанализированы структурные особенности пространства второй когомологии присоединенного представления алгебры Ли $\mathfrak{sl}(2)$. В частности, описано подпространство классов ограниченных коциклов. Доказано, что подпространство классов ограниченных коциклов двумерно и соответствующие локальные деформации являются ограниченными алгебрами Ли в смысе Джекобсона. Выяснено, что неограниченным нетривиальным коциклам соответствует семейство простых трехмерных неограниченных алгебр Ли. В характеристиках 2 и 3 построены трехмерные двусторонние алгебры Алиа, неизоморфные алгебре Ли $\mathfrak{sl}(2)$. В ходе проведенного исследования получено полное описание пространства всех дифференцирований алгебры Ли $\mathfrak{sl}(2)$.

*Ключевые слова:* алгебра Ли, модуль, представление, дифференцирование, внешнее дифференцирование, деформация, ограниченная деформация, когомология, коцикл, коммутативный коцикл, алгебра Алиа.

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# Asymptotic solutions of scalar integro-differential equations with partial derivatives and with rapidly oscillating coefficients

The work is devoted to the development of an asymptotic integration algorithm for the Cauchy problem for a singularly perturbed partial differential integro-differential equation with rapidly oscillating coefficients, which describe various physical processes in micro-inhomogeneous media.This direction in the theory of partial differential equations is developing intensively and finds numerous applications in radiophysics, electrical engineering, filtering theory, phase transition theory, elasticity theory, and other branches of physics, mechanics, and technology. For studies of such processes, asymptotic methods are usually used. It is known that currently rapidly developing numerical methods do not exclude asymptotic. This happens for a number of reasons. Firstly, a reasonably constructed asymptotics, especially its main term, carries information that is important for applications about the qualitative behavior of the solution and, in this sense, to some extent replaces the exact solution, which most often cannot be found. Secondly, as follows from the above, knowledge of the solution structure helps in the development of numerical methods for solving complex problems; therefore, the development of asymptotic methods contributes to the development of numerical methods. Regularization of the problem is carried out, the normal and unique solvability of general iterative problems is proved.

*Keywords:* singularly perturbed, partial integro differential equation, regularization of an integral, solvability of iterative problems.

*Introduction*

A mathematical description of physical processes in micro-inhomogeneousmedia suggests that the local characteristics of the latter depend on a small parameter which is a characteristic scale of the microstructure of the medium. To construct mathematical models of such processes, an asymptotic analysis of the problem is performed for It turns out that the limits of the solutions to the problem are described by some new differential equations that have relatively smoothly varying coefficients and are considered in simple domains.These equations are mathematical models of physical processes in micro-inhomogeneousmedia, and their coefficients are effective characteristics of such media. For mathematical studies of such processes, asymptotic methods are usually used. It is known that currently rapidly developing numerical methods do not exclude asymptotic ones. This happens for a number of reasons. Firstly, a reasonably constructed asymptotics, especially its main term, carries information that is important for applications about the qualitative behavior of the solution and, in this case to some extent replaces the exact solution, which most often cannot be found. Secondly, as follows from the above, knowledge of the solution structure helps in the development of numerical methods for solving complex problems; therefore, the development of asymptotic methods contributes to the development of numerical methods. Thirdly, for some problems, especially those related to fast oscillations, there are simply no effective numerical methods that give a sufficient degree of accuracy. The first of the problems with an irregular dependence in perturbation theory that arose in connection with the problems of celestial mechanics and electrical engineering were nonlinear equations, which are often called oscillating equations at present. Tasks of this kind arise everywhere where certain transient processes take place. Studies of oscillating and singularly perturbed oscillating systems described by ordinary differential equations to the splitting methods were carried out in [1–4] and regularization methods in [5–8]. An analysis of the main results of the study for systems of homogeneous and inhomogeneous differential equations prompted the idea to study singularly perturbed integro-differential equations with rapidly oscillating coefficients.A system of integro-differential equations in the absence of resonance is considered, i.e. when the integer linear combination of frequencies of the rapidly oscillating cosine does not coincide with the frequency of the spectrum of the limit operator [9, 10]. It should be noted that when developing an algorithm for constructing an asymptotic solution to the problem, the ideas of the regularization method used to study ordinary integro-differential equations [11–21] and integro-differential equations with partial derivatives [23–24] were used.

We consider the Cauchy problem for the integro-differential equation with partial derivatives:

$$\begin{array}{l} \varepsilon\frac{\partial y(x,t,\varepsilon)}{\partial x}=a(x)y(x,t,\varepsilon)+\int\limits_{x_0}^{x}K(x,t,s)y(s,t,\varepsilon)ds+h(x,t)+ \\ +\varepsilon g(x)cos\frac{\beta(x)}{\varepsilon}y(x,t,\varepsilon),\quad y(x_0,t,\varepsilon)=y^0(t)\quad ((x,t)\in[x_0,X]\times[0,T]), \end{array} \tag{1}$$

where $\beta'(x)>0$, $g(x)$, $a(x)$ is a scalar functions, $y^0(t)$ constant, $\varepsilon>0$ is a small parameter. Denote by $\lambda_1(x)=-a(x)$, $\beta'(x)$ is a frequency of rapidly oscillating cosine. In the following, functions $\lambda_2(x)=-i\beta'(x)$, $\lambda_3(x)=+i\beta'(x)$ will be called *the spectrum of a rapidly oscillating coefficient.*

We assume that the conditions are fulfilled:

(i) $a(x),g(x),\beta(x)\in C^\infty[x_0,X]$; $h(x,t)\in C^\infty[x_0,X]\times[0,T]$, the kernel $K(x,t,s)$ belongs to the space $K(x,t,s)\in C^\infty\{x_0<x<s<X,\ 0<t<T\}$;

(ii) $\lambda_1(x)\equiv a(x)\neq\lambda_j(x),\quad j=2,3,\quad \lambda_i(x)\neq 0,\ (\forall x\in[x_0,X]),\ i=1,2,3;$

(iii) $\lambda_1(x)\leq 0,\quad (\forall x\in[x_0,X]);$

(iv) for $\forall x\in[x_0,X]$ and $n_2\neq n_3$ inequalities

$$\begin{array}{l} n_2\lambda_2(x)+n_3\lambda_3(x)\neq\lambda_1(x), \\ \lambda_1(x)+n_2\lambda_2(x)+n_3\lambda_3(x)\neq\lambda_1(x),\quad (\forall x\in[x_0,X]) \end{array}$$

for all multi-indices $n=(n_2,n_3)$ with $|n|\equiv n_2+n_3\geq 1$ ($n_2$ and $n_3$ are non-negative integers) are holds.

We will develop an algorithm for constructing a regularized [5] asymptotic solution of problem (1).

### *1 Regularization of problem (1)*

Denote by $\sigma_j=\sigma_j(\varepsilon)$, independent of $t$ magnitudes $\sigma_1=e^{-\frac{i}{\varepsilon}\beta(t_0)}$, $\sigma_1=e^{+\frac{i}{\varepsilon}\beta(t_0)}$, and rewrite system (1) as

$$\begin{array}{l} \varepsilon\frac{\partial y(x,t,\varepsilon)}{\partial x}-\lambda_1(x)y(x,t,\varepsilon)-\varepsilon\frac{g(t)}{2}\left(e^{-\frac{i}{\varepsilon}\int_{t_0}^{t}\beta'(\theta)d\theta}\sigma_1+e^{+\frac{i}{\varepsilon}\int_{t_0}^{t}\beta'(\theta)d\theta}\sigma_2\right)y(x,t,\varepsilon)- \\ -\int_{x_0}^{x}K(x,t,s)\,y(s,t,\varepsilon)ds=h(x,t),\ y(x_0,t,\varepsilon)=y^0,\ ((x,t)\in[x_0,X]\times[0,T]). \end{array} \tag{2}$$

We introduce regularizing variables (see [5, 6]):

$$\tau_j=\frac{1}{\varepsilon}\int_{x_0}^{x}\lambda_j(\theta)\,d\theta\equiv\frac{\psi_j(x)}{\varepsilon},j=\overline{1,3},$$

and instead of problem (2), consider the problem

$$\begin{array}{l} \varepsilon\frac{\partial\tilde y}{\partial x}+\sum\limits_{j=1}^{3}\lambda_j(t)\frac{\partial\tilde y}{\partial\tau_j}-\lambda_1(x)\tilde y-\varepsilon\frac{g(t)}{2}\left(e^{\tau_2}\sigma_1+e^{\tau_3}\sigma_2\right)\tilde y- \\ -\int_{x_0}^{x}K(x,t,s)\,\tilde y(s,t,\frac{\psi(s)}{\varepsilon},\varepsilon)ds=h(x,t),\ ,\tilde y(x,t,\tau,\varepsilon)|_{x=x_0,\tau=0}=y^0,\ ((x,t)\in[x_0,X]\times[0,T]), \end{array} \tag{3}$$

for the function $\tilde y=\tilde y(x,t,\tau,\varepsilon)$, where is indicated: $\tau=(\tau_1,\tau_2,\tau_3)$, $\psi=(\psi_1,\psi_2,\psi_3)$. It is clear that if $\tilde y=\tilde y(x,t,\tau,\varepsilon)$ is a solution to problem (3), then the vector function $y=\tilde y\left(x,\frac{\psi(x)}{\varepsilon},\varepsilon\right)$ is an exact solution to problem (2), therefore, problem (3) is extended with respect to problem (2). However, it cannot be considered fully regularized, since it does not regularize the integral term $J\tilde y=\int_{x_0}^{x}K(x,t,s)\,\tilde y(s,\frac{\psi(s)}{\varepsilon},\varepsilon)ds$. To regularize the integral operator, we introduce a class $M_\varepsilon$ that is asymptotically invariant with respect to the operator $J\tilde y$ (see [5], p. 62). Recall the corresponding concept.

*Definition 1.* A class $M_\varepsilon$ is said to be asymptotically invariant (with $\varepsilon\to+0$) with respect to an operator $P_0$ if the following conditions are fulfilled:

1) $M_\varepsilon\subset D(P_0)$ with each fixed $\varepsilon>0$;

2) the image $P_0g(x,\varepsilon)$ of any element $g(x,\varepsilon)\in M_\varepsilon$ decomposes in a power series

$$P_0g(x,\varepsilon)=\sum_{n=0}^{\infty}\varepsilon^n g_n(x,\varepsilon)(\varepsilon\to+0,g_n(x,\varepsilon)\in M_\varepsilon,n=0,1,...),$$

convergent asymptotically for $\varepsilon\to+0$ (uniformly with $x\in[x_0,X]$ ).

From this definition it can be seen that the class $M_\varepsilon$ depends on the space $U$, in which the operator $P_0$ is defined. In our case $P_0 = J$. For the space $U$ we take the space of vector functions $y\,(x,t,\tau)$, represented by sums

$$y\,(x,t,\tau,\sigma)=y_0\,(x,t,\sigma)+\sum_{i=1}^{3}y_i\,(x,t,\sigma)\,e^{\tau_i}+\sum_{2\le|m|\le N_y}^{*}y^m\,(x,t,\sigma)\,e^{(m,\tau)}+$$

$$+\sum_{1\le|m|\le N_y}^{*}y^{e_1+m}\,(x,t,\sigma)\,e^{(e_1+m,\tau)},y_i\,(x,t,\sigma)\,,y^m\,(x,t,\sigma)\,,y^{e_1+m}\,(x,t,\sigma)\in C^\infty[x_0,X]\times[0,T],\qquad(4)$$

$$m=(0,m_2,m_3)\,,1\le|m|\equiv m_2+m_3\le N_y,\,i=\overline{1,3},$$

where is denoted: $\lambda\,(x)\equiv(\lambda_1,\lambda_2,\lambda_3)\,,\ (m,\lambda\,(x))\equiv m_2\lambda_2\,(x)+m_3\lambda_3\,(x)\,,\,(e_1+m,\lambda\,(x))\equiv\lambda_1\,(x)+m_2\lambda_3\,(x)+$ $+m_3\lambda_3\,(x)$; an asterisk $*$ above the sum sign indicates that the summation for $|m|\ge1$ it occurs only over multi-indices $m=(0,m_2,m_3)$ with $m_2\ne m_3$, $e_1=(1,0,0)\,,\,\sigma=(\sigma_1,\sigma_2)\,.$

Note that here the degree $N_y$ of the polynomial $y\,(x,t,\tau,\sigma)$ relative to the exponentials $e^{\tau_1}$ depends on the element $y$. In addition, the elements of space $U$ depend on bounded in $\varepsilon>0$ terms of constants $\sigma_1=\sigma_1\,(\varepsilon)$ and $\sigma_2=\sigma_2\,(\varepsilon)$, and which do not affect the development of the algorithm described below, therefore, in the record of element (4) of this space $U$, we omit the dependence on $\sigma=(\sigma_1,\sigma_2)$ for brevity. We show that the class $M_\varepsilon=U|_{\tau=\psi(t)/\varepsilon}$ is asymptotically invariant with respect to the operator $J$. The image of the operator on the element (4) of the space $U$ has the form

$$Jy(x,t,\tau)=\int_{x_0}^{x}K(x,t,s)y_0(s,t)ds+\sum_{i=1}^{3}\int_{x_0}^{x}K(x,t,s)y_i(s,t)e^{\frac{1}{\varepsilon}\int_{x_0}^{s}\lambda_i(\theta)d\theta}ds+$$

$$+\sum_{2\le|m|\le N_z}^{*}\int_{x_0}^{x}K(x,t,s)y^m(s,t)e^{\frac{1}{\varepsilon}\int_{x_0}^{s}(m,\lambda(\theta))d\theta}ds+\sum_{1\le|m|\le N_z}^{*}\int_{x_0}^{x}K\,(x,t,s)\,y^{e_1+m}\,(s,t)\,e^{\frac{1}{\varepsilon}\int_{x_0}^{s}(e_1+m,\lambda(\theta))d\theta}ds.$$

Integrating in parts, we will have

$$J_i(x,t,\varepsilon)=\int_{x_0}^{x}K(x,t,s)y_i(s,t)e^{\frac{1}{\varepsilon}\int_{x_0}^{s}\lambda_i(\theta)d\theta}ds=\varepsilon\int_{x_0}^{x}\frac{K(x,t,s)y_i(s,t)}{\lambda_i(s)}de^{\frac{1}{\varepsilon}\int_{x_0}^{s}\lambda_i(\theta)d\theta}=$$

$$=\varepsilon\left[\frac{K(x,t,s)y_i(s,t)}{\lambda_i(s)}e^{\frac{1}{\varepsilon}\int_{x_0}^{s}\lambda_i(\theta)d\theta}\Bigg|_{s=x_0}^{s=x}-\int_{x_0}^{x}\left(\frac{\partial}{\partial s}\frac{K(x,t,s)y_i(s,t)}{\lambda_i(\theta)}\right)e^{\frac{1}{\varepsilon}\int_{x_0}^{s}\lambda_i(\theta)d\theta}ds\right]=$$

$$=\varepsilon\left[\frac{K(x,t,x)y_i(x,t)}{\lambda_i(x)}e^{\frac{1}{\varepsilon}\int_{x_0}^{x}\lambda_i(\theta)d\theta}-\frac{K(x,t,x_0)y_i(x_0,t)}{\lambda_i(x_0)}\right]-\varepsilon\int_{x_0}^{x}\left(\frac{\partial}{\partial s}\frac{K(x,t,s)y_i(s,t)}{\lambda_i(s)}\right)e^{\frac{1}{\varepsilon}\int_{x_0}^{s}\lambda_i(\theta)d\theta}ds\,.$$

Continuing this process further, we obtain the decomposition

$$J_i(x,t,\varepsilon)=\sum_{\nu=0}^{\infty}(-1)^\nu\varepsilon^{\nu+1}\left[\left(I_i^\nu\left(K(x,t,s)y_i(s,t)\right)\right)_{s=x}e^{\frac{1}{\varepsilon}\int_{x_0}^{x}\lambda_i(\theta))\,d\theta}-\left(I_i^\nu\left(K(x,t,s)y_i(s,t)\right)\right)_{s=x_0}\right],$$

$I_i^0=\frac{1}{\lambda_i(s)}\cdot,I_i^\nu=\frac{1}{\lambda_i(s)}I_i^{\nu-1}(\nu\ge1,\,i=\overline{1,3}).$

Applying the integration operation in parts to integrals

$$J_m(x,t,\varepsilon)=\int_{x_0}^{x}K(x,t,s)y^m(s,t)e^{\frac{1}{\varepsilon}\int_{x_0}^{s}(m,\lambda(\theta))d\theta}ds,$$

$$J_{e_1+m}(x,t,\varepsilon)=\int_{x_0}^{x}K(x,t,s)y^{e_1+m}\,(s,t)\,e^{\frac{1}{\varepsilon}\int_{x_0}^{s}(e_1+m,\lambda(\theta))d\theta}ds,$$

we note that for all multi-indices $m=(0,m_2,m_3)\,,m_2\ne m_3$, inequalities

$$(m,\lambda(x))\equiv m_2\lambda_2(x)+m_3\lambda_3(x)\ne0\quad\forall x\in[x_0,X]\,,\ \ m_2+m_3\ge2.$$

are satisfied. In addition, for the same multi-indices $m=(0,m_2,m_3)$ we have

$$(e_1+m,\lambda(x))\neq 0\ \forall x\in[x_0,X]\,,\ m_2\neq m_3,\ |m|=m_2+m_3\geq 1.$$

Indeed, if $(e_1+m,\lambda(x))=0$ for some $x\in[x_0,X]$ and $m_2\neq m_3$, $m_2+m_3\geq 1$, then $m_2\lambda_2(x)+m_3\lambda_3(x)= =-\lambda_1(x)$, $m_2+m_3\geq 1$, which contradicts condition (iv). Therefore, integration by parts in integrals is possible. Performing it, we will have:

$$J_m(x,t,\varepsilon)=\int\limits_{t_0}^{x}K(x,t,s)y^m(s,t)e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{s}(m,\lambda(\theta))d\theta}ds=\varepsilon\int\limits_{x_0}^{x}\frac{K(x,t,s)y^m(s,t)}{(m,\lambda(s))}de^{\frac{1}{\varepsilon}\int\limits_{x_0}^{s}(m,\lambda(\theta))d\theta}=$$

$$=\varepsilon\left[\frac{K(x,t,s)y^m(x,t)}{(m,\lambda(s))}e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{s}(m,\lambda(\theta))d\theta}\Bigg|_{s=x_0}^{s=x}-\varepsilon\int\limits_{x_0}^{x}\left(\frac{\partial}{\partial s}\frac{K(x,t,s)y^m(s,t)}{(m,\lambda(s))}\right)e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{s}(m,\lambda(\theta))d\theta}\right]-$$

$$=\varepsilon\left[\frac{K(x,t,x)y^m(x,t)}{(m,\lambda(x))}e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{x}(m,\lambda(\theta))d\theta}-\frac{K(x,t,x_0)y^m(x_0,t)}{(m,\lambda(x_0))}\right]-$$

$$-\varepsilon\int\limits_{x_0}^{x}\left(\frac{\partial}{\partial s}\frac{K(x,t,s)y^m(s,t)}{(m,\lambda(s))}\right)e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{s}(m,\lambda(\theta))d\theta}ds.$$

Therefore, the image of the operator $J$ on the element (4) of the space $U$ is represented as a series

$$Jz\left(t,\tau\right)=\int_{t_0}^{t}K\left(t,s\right)z_0\left(s\right)ds+\sum_{i=1}^{4}\sum_{\nu=0}^{\infty}(-1)^{\nu}\varepsilon^{\nu+1}[(I_i^{\nu}\left(K\left(t,s\right)z_i\left(s\right)\right))_{s=t}e^{\frac{1}{\varepsilon}\int_{t_0}^{t}\lambda_i(\theta))d\theta}-$$
$$-\left(I_i^{\nu}\left(K\left(t,s\right)z_i\left(s\right)\right)\right)_{s=t_0}]+\sum_{\nu=0}^{\infty}(-1)^{\nu}\varepsilon^{\nu+1}[(I_m^{\nu}\left(K\left(t,s\right)z^m\left(s\right)\right))_{s=t}e^{\frac{1}{\varepsilon}\int_{t_0}^{t}(m,\lambda(\theta))d\theta}-$$
$$-\left(I_m^{\nu}\left(K\left(t,s\right)z^m\left(s\right)\right)\right)_{s=t_0}]+\sum_{j=1}^{2}\sum_{1\leq|m|\leq N_z}\sum_{\nu=0}^{\infty}(-1)^{\nu}\varepsilon^{\nu+1}[\left(I_{j,m}^{\nu}\left(K\left(t,s\right)z^{e_j+m}\left(s\right)\right)\right)_{s=t}\times$$
$$\times e^{\frac{1}{\varepsilon}\int_{t_0}^{t}(e_j+m,\lambda(\theta))}-\left(I_{j,m}^{\nu}\left(K\left(t,s\right)z^{e_j+m}\left(s\right)\right)\right)_{s=t_0}]_{\tau=\psi(t)/\varepsilon}.$$

Continuing this process, we obtain the series

$$J_m(x,t,\varepsilon)=\sum_{\nu=0}^{\infty}(-1)^{\nu}\varepsilon^{\nu+1}\Bigg[(I_m^{\nu}\left(K(x,t,s)y^m(s,t)\right))_{s=t}e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{x}(m,\lambda(\theta))d\theta}-$$
$$-\left(I_m^{\nu}\left(K(x,t,s)y^m(s,t)\right)\right)_{s=t_0}\Big],$$

$$I_m^0=\frac{1}{(m,\lambda(s))}.,I_m^{\nu}=\frac{1}{(m,\lambda(s))}\frac{\partial}{\partial s}I_m^{\nu-1}\,(\nu\geq 1,|m|\geq 2),$$

$$J_{e_1+m}(x,t,\varepsilon)=\int\limits_{x_0}^{x}K(x,t,s)y^{e_1+m}(s,t)e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{s}(e_1+m,\lambda(\theta))d\theta}ds=$$
$$=\varepsilon\int\limits_{x_0}^{s}\frac{K(x,t,s)y^{e_1+m}(s,t)}{(e_1+m,\lambda(s))}de^{\frac{1}{\varepsilon}\int\limits_{x_0}^{s}(e_1+m,\lambda(\theta))d\theta}=$$
$$=\varepsilon\left[\frac{K(x,t,s)y^{e_1+m}(s,t)}{(e_1+m,\lambda(s))}e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{s}(e_1+m,\lambda(\theta))d\theta}\Bigg|_{s=x_0}^{s=x}-\right.$$
$$\left.-\varepsilon\int\limits_{x_0}^{x}\left(\frac{\partial}{\partial s}\frac{K(x,t,s)y^{e_1+m}(s,t)}{(e_1+m,\lambda(s))}\right)e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{s}(e_1+m,\lambda(\theta))d\theta}ds\right]-$$
$$=\varepsilon\left[\frac{K(x,t,x)y^{e_1+m}\,(x,t)}{(e_1+m,\lambda(x))}e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{x}(e_1+m,\lambda(\theta))d\theta}-\frac{K(x,t,x_0)y^{e_1+m}\,(x_0,t)}{(e_1+m,\lambda(x_0))}\right]-$$

$$-\varepsilon\int\limits_{x_0}^{x}\left(\frac{\partial}{\partial s}\frac{K(x,t,s)y^{e_1+m}(s,t)}{(e_1+m,\lambda(s))}\right)e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{s}(e_1+m,\lambda(\theta))d\theta}ds.$$

Continuing this process, we obtain the series

$$J_{e_1+m}(x,t,\varepsilon)=\sum_{\nu=0}^{\infty}(-1)^{\nu}\varepsilon^{\nu+1}\left[\left(I^{\nu}_{e_1+m}\left(K(x,t,s)y^{e_1+m}(s,t)\right)\right)_{s=x}e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{x}(e_1+m,\lambda(\theta))d\theta},\right.$$
$$\left.-\left(I^{\nu}_{e_1+m}\left(K(x,t,s)y^{e_1+m}(s,t)\right)\right)_{s=x_0}\right],$$

$$I^0_{e_1+m}=\frac{1}{(e_1+m,\lambda(s))}.,I^{\nu}_{j,m}=\frac{1}{(e_1+m,\lambda(s))}\frac{\partial}{\partial s}I^{\nu-1}_m(\nu\geq 1,\,|m|\geq 1),$$

Therefore, the image of the operator $j$ on the element (4) of the space $U$ is represented as a series

$$Jy(x,t,\tau)=\int\limits_{x_0}^{x}K(x,t,s)y_0(s,t)ds+\sum_{i=1}^{3}\sum_{\nu=0}^{\infty}(-1)^{\nu}\varepsilon^{\nu+1}\left[\left(I^{\nu}_i\left(K(x,t,s)y_i(s,t)\right)\right)_{s=t}e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{x}\lambda_i(\theta))\,d\theta}-\right.$$
$$\left.-\left(I^{\nu}_i\left(K(x,t,s)y_i(s,t)\right)\right)_{s=t_0}\right]+$$
$$+\sum^{*}_{2\leq|m|\leq N_Y}\sum_{\nu=0}^{\infty}(-1)^{\nu}\varepsilon^{\nu+1}\left[\left(I^{\nu}_m\left(K(x,t,s)y^m(s,t)\right)\right)_{s=t}e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{x}(m,\lambda(\theta))\,d\theta}-\left(I^{\nu}_m\left(K(x,t,s)y^m(s,t)\right)\right)_{s=t_0}\right]+$$
$$+\sum_{1\leq|m|\leq N_Y}\sum_{\nu=0}^{\infty}(-1)^{\nu}\varepsilon^{\nu+1}\left[\left(I^{\nu}_{e_1+m}\left(K(x,t,s)y^{e_1+m}(s,t)\right)\right)_{s=x}e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{x}(e_1+m,\lambda(\theta))d\theta}-\right.$$
$$\left.-\left(I^{\nu}_{e_1+m}\left(K(x,t,s)y^{e_1+m}(s,t)\right)\right)_{s=x_0}\right].$$

It is easy to show (see, for example, [25], pp. 291-294) that this series converges asymptotically for $\varepsilon\to+0$ (uniformly in $(x,t)\in[x_0,X]\times[0,T]$). This means that the class $M_\varepsilon$ is asymptotically invariant (for $\varepsilon\to+0$) with respect to the operator $J$.

We introduce operators $R_\nu:\ U\to U$, acting on each element $y(x,t,\tau)\in U$ of the form (4) according to the law:

$$R_0y\,(x,t,\tau)=\int\limits_{x_0}^{x}K(x,t,s)y_0(s,t)ds, \tag{$5_0$}$$

$$R_1y(x,t,\tau)=\sum_{i=1}^{3}\left[\left(I^0_i\left(K(x,t,s)y_i(s,t)\right)\right)_{s=x}e^{\tau_i}-\left(I^0_i\left(K(x,t,s)y_i(s,t)\right)\right)_{s=x_0}\right]+$$
$$+\sum^{*}_{1\leq|m|\leq N_z}\left[\left(I^0_m\left(K(x,t,s)y^m\,(s,t)\right)\right)_{s=x}e^{(m,\tau)}-\left(I^0_m\left(K(x,t,s)y^m\,(s,t)\right)\right)_{s=x_0}\right]+ \tag{$5_1$}$$
$$+\sum^{*}_{1\leq|m|\leq N_z}\left[\left(I^0_{e_1+m}\left(K(x,t,s)y^{e_1+m}\,(s,t)\right)\right)_{s=x}e^{(e_1+m,\tau)}-\left(I^0_{e_1+m}\left(K(x,t,s)y^{e_1+m}\,(s,t)\right)\right)_{s=x_0}\right],$$

$$R_{\nu+1}y(x,t,\tau)=\sum_{i=1}^{3}\left[\left(I^{\nu}_i\left(K(x,t,s)y_i\,(s,t)\right)\right)_{s=x}e^{\tau_i}-\left(I^{\nu}_i\left(K(x,t,s)y_i\,(s,t)\right)\right)_{s=x_0}\right]+$$
$$+\sum^{*}_{2\leq|m|\leq N_y}\left[\left(I^{\nu}_m\left(K(x,t,s)y^m\,(s,t)\right)\right)_{s=x}e^{(m,\tau)}-\left(I^{\nu}_m\left(K(x,t,s)y^m\,(s,t)\right)\right)_{s=x_0}\right]+ \tag{$5_{\nu+1}$}$$
$$+\sum^{*}_{1\leq|m|\leq N_z}\left[\left(I^{\nu}_{e_1+m}\left(K(x,t,s)z^{e_1+m}\,(s,t)\right)\right)_{s=x}e^{(e_1+m,\tau)}-\left(I^{\nu}_{e_1+m}\left(K(x,t,s)z^{e_1+m}\,(s,t)\right)\right)_{s=x_0}\right],\nu\geq 1.$$

Now let $\tilde{y}(x,t,\tau,\varepsilon)$ be an arbitrary continuous function on $(x,t,\tau) \in [x_0,X] \times [0,T] \times \{Re\lambda_1(x)\}$ with asymptotic expansion

$$\tilde{y}(x,t,\tau,\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k y_k(x,t,\tau), \quad y_k(x,t,\tau) \in U, \tag{6}$$

converging as $\varepsilon \to +0$ (uniformly in $(x,t,\tau) \in [x_0,X] \times [0,T] \times \{Re\lambda_1(x)\}$). Then the image $J\tilde{y}(x,t,\tau,\varepsilon)$ of this function is decomposed into an asymptotic series

$$J\tilde{z}(x,t,\tau,\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k J y_k(x,t,\tau) = \sum_{r=0}^{\infty} \varepsilon^r \sum_{s=0}^{r} R_{r-s} y_s(x,t,\tau)\,|_{\tau=\psi(x)/\varepsilon}.$$

This equality is the basis for introducing an extension of an operator $J$ on series of the form (6):

$$\tilde{J}\tilde{y}(x,t,\tau,\varepsilon) \equiv \tilde{J}\left(\sum_{k=0}^{\infty} \varepsilon^k y_k(x,t,\tau)\right) \triangleq \sum_{r=0}^{\infty} \varepsilon^r \sum_{s=0}^{r} R_{r-s} y_s(x,t,\tau).$$

Although the operator $\tilde{J}$ is formally defined, its utility is obvious, since in practice it is usual to construct the $N$–th approximation of the asymptotic solution of the problem (2), in which impose only $N$-th partial sums of the series (6), which have not a formal, but a true meaning. Now you can write a problem that is completely regularized with respect to the original problem (2):

$$\begin{gathered} L_\varepsilon \tilde{y}(x,t,\tau,\varepsilon) \equiv \varepsilon \tfrac{\partial \tilde{y}}{\partial x} + \sum_{j=1}^{3} \lambda_j(x) \tfrac{\partial \tilde{y}}{\partial \tau_j} - \lambda_1(x)\tilde{y} - \tilde{J}\tilde{y} - \varepsilon \tfrac{g(x)}{2}(e^{\tau_2}\sigma_1 + e^{\tau_3}\sigma_2)\tilde{y} = \\ = h(x,t), \quad \tilde{y}(x_0,t,0,\varepsilon) = y^0, \quad ((x,t) \in [x_0,X] \times [0,T]). \end{gathered} \tag{7}$$

*2 Solvability of iterative problems*

Substituting the series (6) into (7) and equating the coefficients with the same degrees $\varepsilon$, we obtain the following iterative problems:

$$Ly_0(x,t,\tau) \equiv \sum_{j=1}^{3} \lambda_j(x) \frac{\partial y_0}{\partial \tau_j} - \lambda_1(x) y_0 - R_0 y_0 = h(x,t),\ y_0(x_0,t,0) = y^0(t); \tag{$8_0$}$$

$$Ly_1(x,t,\tau) = -\frac{\partial y_0}{\partial x} + \frac{g(x)}{2}(e^{\tau_2}\sigma_1 + e^{\tau_3}\sigma_2) y_0 + R_1 y_0,\ y_1(x_0,t,0) = 0; \tag{$8_1$}$$

$$Ly_2(x,t,\tau) = -\frac{\partial y_1}{\partial x} + \frac{g(x)}{2}(e^{\tau_2}\sigma_1 + e^{\tau_3}\sigma_2) y_1 + R_1 y_1 + R_2 y_0,\ y_0(x_0,t,0) = 0; \tag{$8_2$}$$

$$\ldots$$

$$Ly_k(x,t,\tau) = -\frac{\partial y_{k-1}}{\partial x} + \frac{g(x)}{2}(e^{\tau_2}\sigma_1 + e^{\tau_3}\sigma_2) y_{k-1} + R_k y_0 + ... + R_1 y_{k-1},\ \ y_k(x_0,t,0) = 0,\ k \geq 1. \tag{$8_k$}$$

Each of the iterative problems $(8_k)$ can be written as

$$Lz(x,t,\tau) \equiv \sum_{j=1}^{3} \lambda_j(x) \frac{\partial y}{\partial \tau_j} - \lambda_1(x) y - R_0 y = h(x,t,\tau),\ y(x_0,t,0) = y^*, \tag{9}$$

where

$$h(x,t,\tau) = h_0(x,t) + \sum_{i=1}^{3} h_i(x,t) e^{\tau_i} + \sum_{2 \leq |m| \leq N_y}^{*} h^m(x,t) e^{(m,\tau)} + \sum_{1 \leq |m| \leq N_y}^{*} h^{e_1+m}(x,t) e^{(e_1+m,\tau)} \in U,$$

is the known vector function of space $U$, $y^*$ is the known constant vector of the complex space $C$, and the operator $R_0$ has the form (see $(5_0)$)

$$R_0 y(x,t,\tau) \equiv R_0 \left[ y_0(x,t) + \sum_{i=1}^{3} y_i(x,t) e^{\tau_i} + \sum_{2 \leq |m| \leq N_y}^{*} y^m(x,t) e^{(m,\tau)} + \sum_{1 \leq |m| \leq N_y}^{*} y^{e_1+m}(x,t) e^{(e+m,\tau)} \right] \triangleq$$

$$\triangleq \int_{x_0}^{x} K(x,t,s) y_0(s,t) ds.$$

We will determine the solution of equation (9) as an element (4) of the space $U$:

$$y(x,t,\tau)=y_0(x,t)+\sum_{i=1}^{3}y_i(x,t)e^{\tau_i}+\sum_{2\le|m|\le N_z}^{*}y^m(x,t)e^{(m,\tau)}+\sum_{1\le|m|\le N_H}^{*}y^{e_1+m}(x,t)e^{(e_1+m,\tau)}\equiv$$

$$\equiv y_0(x,t)+\sum_{i=1}^{3}y_i(x,t)e^{\tau_i}+\sum_{2\le|m|\le N_y}^{*}y^m(x,t)e^{(m,\tau)}+\sum_{2\le|m^1|\le N_H}^{*}y^{m^1}(x,t)e^{(m^1,\tau)}, \quad (10)$$

where for convenience introduced multi-indices

$$m^1=e_1+m\equiv(1,m_2,m_3)\,,\ \left|m^1\right|=1+m_2+m_3\ge 2,$$

$m_2$ and $m_3$ are non-negative integer numbers. Substituting (10) into equation (9), and equating here the free terms and coefficients separately for identical exponents, we obtain the following equations:

$$-\lambda_1(x)y_0(x,t)-\int\limits_{x_0}^{x}K(x,t,s)y_0(s,t)ds=h_0(x,t), \quad (11)$$

$$\left[\lambda_i(x)-\lambda_1(x)\right]y_i(x,t)=h_i(x,t),\quad i=\overline{1,3}, \quad (11_i)$$

$$\left[(m,\lambda(x))-\lambda_1(x)\right]y^m(x,t)=h^m(x,t),\quad m_2\ne m_3,\quad 2\le|m|\le N_h, \quad (11_m)$$

$$\left[\left(m^1,\lambda(x)\right)-\lambda_1(x)\right]y^{m^1}(x,t)=h^{m^1}(x,t),\quad m_2\ne m_{3,}\ \ 1\le\left|m^1\right|\le N_h. \quad (12)$$

The equation (11) can be written as

$$y_0(x,t)=\int\limits_{t_0}^{t}\left(-\lambda_1^{-1}(x)K(x,t,s)\right)y_0(s,t)ds-\lambda_1^{-1}(x)h_0(x,t). \quad (11_0)$$

Due to the smoothness of the kernel $-\lambda_1^{-1}(x)K(x,t,s)$ and heterogenei ty $-\lambda_1^{-1}(x)h_0(x,t)$, this Volterra integral system has a unique solution $y_0(x,t)\in C^\infty[x_0,X]\times[0,T]$. The equations $(11_2)$ - $(11_3)$ also have unique solutions

$$y_i(x,t)=\left[\lambda_i(x)-\lambda_1(x)\right]^{-1}h_i(x,t)\in C^\infty[x_0,X]\times[0,T],\ \ i=2,3,$$

since $\lambda_i(x)\ne\lambda_1(x)$, $i=2,3$. Equation $(11_1)$ are solvable in space $C^\infty[x_0,X]\times[0,T]$ if and only

$$h_1(x,t)\equiv 0\ \ (\forall(x,t)\in[x_0,X]\times[0,T])\,. \quad (13)$$

Further, since $(m,\lambda(x))\equiv m_2\lambda_2(x)+m_3\lambda_3(x)\ne\lambda_1(x),\ \ |m|=m_2+m_3\ge 2$ (see condition (iv) the absence of resonance), the equation $(11_m)$ has a unique solution

$$y^m(x,t)=\left[(m,\lambda(x))-\lambda_1(x)\right]^{-1}h^m(x,t)\in C^\infty[x_0,X]\times[0,T],\ \ 2\le|m|\le N_h.$$

We now equation (12). Let us show that when $\left|m^1\right|\ge 1$ the functions $\left(m^1,\lambda(x)\right)\ne\lambda_1(x)$. Indeed, let $\left(m^1,\lambda(t)\right)=\lambda_1(t),\quad\left|m^1\right|\ge 1$. Then

$$\lambda_1(x)+m_2\lambda_2(x)+m_3\lambda_3(x)=\lambda_1(x)\ \Leftrightarrow\ m_2\lambda_2(x)+m_3\lambda_3(x)=0\ \Leftrightarrow\ m_2\ne m_3,\ \ m_2+m_3\ge 1,$$

which cannot be (see definition of class $U$). Thus, equation (12) for $\left|m^1\right|\ge 1$ has a unique solution

$$z^{m^1}(x,t)=\left[\left(m^1,\lambda(x)\right)-\lambda_1(x)\right]^{-1}h^{m^1}(x,t),\ \ 1\le\left|m^1\right|\le N_h,$$

inn class $C^\infty[x_0,X]\times[0,T]$.

We have proved the following statement.

*Theorem 1. Let conditions (i)-(ii), (iv) be fulfilled and the right-hand side*

$$h(x,t,\tau)=h_0(x,t)+\sum_{i=1}^{3}h_i(x,t)e^{\tau_i}+\sum_{2\le|m|\le N_z}^{*}h^m(x,t)e^{(m,\tau)}+\sum_{1\le|m|\le N_H}^{*}h^{e_1+m}(x,t)e^{(e_1+m,\tau)}\in U$$

*of equation (9) belongs to the space $U$. Then, for the solvability of equation (9) in space $U$, it is necessary and sufficient that condition (13) is satisfied.*

*Under constraint (13), equation (9) has the following solution in space $U$:*

$$y(x,t,\tau)=y_0(x,t)+\xi(x,t)e^{\tau_1}+\sum_{i=2}^{3}\left[\lambda_i(x)-\lambda_1(x)\right]^{-1}H_i(x,t)e^{\tau_i}+$$

$$+\sum_{2\le|m|\le N_H}^{*}\left[(m,\lambda(x))-\lambda_1(x)\right]^{-1}H^m(x,t)e^{(m,\tau)}+ \tag{14}$$

$$\sum_{1\le|m|\le N_H}^{*}\left[\left(m^1,\lambda(x)\right)-\lambda_1(x)\right]^{-1}H^{e_1+m}(x,t)e^{(e_1+m,\tau)},$$

*where $\xi(x,t)\in C^\infty[x_0,X]\times[0,T]$ are arbitrary function, $y_0(x,t)$ is the solution of an integral equation $(11_0)$,* $m\equiv(0,m_2,m_3)\ ,m_2\neq m_3,\ |m|=m_2+m_3\geq 1$.

Subject the solution (14) to the initial condition $y(x_0,t,0)=y_*(t)$. Then we have

$$\xi(x_0,t)-\lambda_1^{-1}(x_0)h_0(x_0,t)+\sum_{i=2}^{3}[\lambda_i(x_0)-\lambda_1(x_0)]^{-1}h_i(x_0,t)+$$

$$+\sum_{2\le|m|\le N_h}^{*}\left[(m,\lambda(x_0))-\lambda_1(x_0)\right]^{-1}h^m(x_0,t)+\sum_{1\le|m|\le N_h}^{*}\left[\left(m^1,\lambda(x_0)\right)-\lambda_1(x_0)\right]^{-1}h^{e_1+m}(x_0,t)=y_*\Leftrightarrow$$

$$\Leftrightarrow\ \ \xi(x_0,t)=y_*+\lambda_1^{-1}(x_0)h_0(x_0,t)-\sum_{i=2}^{3}[\lambda_i(x_0)-\lambda_1(x_0)]^{-1}h_i(x_0,t)- \tag{15}$$

$$-\sum_{2\le|m|\le N_h}^{*}\left[(m,\lambda(x_0))-\lambda_1(x_0)\right]^{-1}h^m(x_0,t)-\sum_{1\le|m|\le N_h}^{*}\left[\left(m^1,\lambda(x_0)\right)-\lambda_1(x_0)\right]^{-1}h^{e_1+m}(x_0,t).$$

However, the functions $\xi(x,t)$ were not found completely. An additional requirement is required to solve problem (13). Such a requirement is dictated by iterative problems $(8_k)$, from which it can be seen that the natural additional constraint is the condition

$$-\frac{\partial y}{\partial x}+\frac{g(x)}{2}(e^{\tau_2}+e^{\tau_3})y+R_1y+p(x,t,\tau)\equiv 0,\qquad (\forall(x,t)\in[x_0,X]\times[0,T]), \tag{16}$$

where $p(x,t,\tau)=p_0(x,t)+\sum\limits_{i=1}^{3}p_i(x,t)e^{\tau_i}+\sum\limits_{2\le|m|\le N_z}^{*}p^m(x,t)e^{(m,\tau)}+\sum\limits_{1\le|m|\le N_H}^{*}p^{e_1+m}(x,t)e^{(e_1+m,\tau)}\in U$ is the known vector-function. The right part of this equation:

$$G(x,t,\tau)\equiv-\frac{\partial y}{\partial t}+\frac{g(x)}{2}\left(e^{\tau_2}\sigma_1+e^{\tau_3}\sigma_2\right)y+Q(x,t,\tau)=$$

$$=-\frac{\partial}{\partial x}\left[y_0(x,t)+\sum_{i=1}^{3}y_i(x,t)e^{\tau_i}+\sum_{2\le|m|\le N_y}^{*}y^m(x,t)e^{(m,\tau)}+\sum_{1\le|m|\le N_y}^{*}y^{e_1+m}(x,t)e^{(e_1+m,\tau)}\right]+$$

$$+\frac{g(x)}{2}\left(e^{\tau_2}\sigma_1+e^{\tau_3}\sigma_2\right)\left[y_0(x,t)+\sum_{i=1}^{3}y_i(x,t)e^{\tau_i}+\sum_{2\le|m|\le N_y}^{*}y^m(x,t)e^{(m,\tau)}+\right.$$

$$\left.+\sum_{1\le|m|\le N_y}^{*}y^{e_1+m}(x,t)e^{(e_1+m,\tau)}\right]+p(x,t,\tau),$$

may not belong to space $U$, if $y = y(x,t,\tau) \in U$. Indeed, taking into account the form (14) of the function $y = y(x,t,\tau) \in U$, we will have

$$Z(x,t,\tau) \equiv G(x,t,\tau) + \frac{\partial y}{\partial x} = \frac{g(x)}{2}\left(e^{\tau_2}\sigma_1 + e^{\tau_3}\sigma_2\right)\left[y_0(x,t) + \sum_{i=1}^{3} y_i(x,t)e^{\tau_i} + \right.$$

$$\left. + \sum_{2\le|m|\le N_z}^{*} y^m(x,t)e^{(m,\tau)} + \sum_{1\le|m|\le N_z}^{*} y^{e_1+m}(x,t)e^{(e_j+m,\tau)}\right] =$$

$$= \frac{g(x)}{2} y_0(x,t)\left(e^{\tau_2}\sigma_1 + e^{\tau_3}\sigma_2\right) + \sum_{i=2}^{3}\frac{g(x)}{2} y_i(x,t)\left(e^{\tau_1+\tau_2}\sigma_1 + e^{\tau_1+\tau_3}\sigma_2\right) +$$

$$+ \frac{g(x)}{2}\left(e^{\tau_2}\sigma_1 + e^{\tau_3}\sigma_2\right)\left[\sum_{2\le|m|\le N_z}^{*} y^m(x,t)e^{(m,\tau)} + \sum_{1\le|m|\le N_z}^{*} y^{e_1+m}(x,t)e^{(e_1+m,\tau)}\right] + p(x,t,\tau).$$

Here are terms with exponents

$$e^{\tau_2+\tau_3} = e^{(m,\tau)}|_{m=(0,1,1)},\ \ e^{\tau_2+(m,\tau)}(\ m_2+1=m_3),\ e^{\tau_3+(m,\tau)}(m_3+1=m_2),$$
$$e^{\tau_2+(e_1+m,\tau)}(\ m_2+1=m_3),\ e^{\tau_3+(e_1+m,\tau)}(m_3+1=m_2) \qquad (*)$$

do not belong to space $U$, since in multi-index $m = (0, m_2, m_3)$ of the space $U$ must be $m_2 \neq m_3$, $m_2 + m_3 \geq 1$. Then, according to the well-known theory (see [5], p. 234), we embed these terms in the space $U$ according to the following rule (see $(*)$):

$$\widehat{e^{\tau_3+\tau_2}} = 1,\ \widehat{e^{\tau_2+(m,\tau)}} = 1\,(m_2+1=m_3,\ m_2\neq m_3),\ \widehat{e^{\tau_3+(m,\tau)}} = 1\ (m_3+1=m_2, m_2\neq m_3),$$
$$\widehat{e^{\tau_2+(e_1+m,\tau)}} = e^{\tau_1}(m_2+1=m_3, m_2\neq m_3),\ \widehat{e^{\tau_3+(e_1+m,\tau)}} = e^{\tau_2}(m_3+1=m_2, m_2\neq m_3). \qquad (**)$$

In $Z(x,t,\tau)$ need of embedding only the terms

$$M(x,t,\tau) \equiv \sum_{i=2}^{3}\frac{g(x)}{2} y_i(x,t)\left(e^{\tau_i+\tau_2}\sigma_1 + e^{\tau_i+\tau_3}\sigma_2\right) + \frac{g(x)}{2} y_1(x,t)\left(e^{\tau_1+\tau_2}\sigma_1 + e^{\tau_1+\tau_3}\sigma_2\right),$$

$$S(x,t,\tau) \equiv \frac{g(x)}{2}\left(e^{\tau_2}\sigma_1 + e^{\tau_3}\sigma_2\right)\left[\sum_{2\le|m|\le N_z}^{*} y^m(x,t)e^{(m,\tau)} + \sum_{1\le|m|\le N_z}^{*} y^{e_1+m}(x,t)e^{(e_1+m,\tau)}\right].$$

We describe this embedding in more detail, taking into account formulas $(**)$ :

$$M(x,t,\tau) \equiv \sum_{k=1}^{2}\frac{g(x)}{2} y_k(x,t)\left(e^{\tau_k+\tau_2}\sigma_1 + e^{\tau_k+\tau_3}\sigma_2\right) + \frac{g(x)}{2} y_1(x,t)\left(e^{\tau_1+\tau_2}\sigma_1 + e^{\tau_1+\tau_3}\sigma_2\right) =$$

$$= \frac{g(x)}{2}\left[y_1(x,t)e^{\tau_1+\tau_2}\sigma_1 + y_1(x,t)e^{\tau_1+\tau_3}\sigma_2 + y_2(x,t)e^{2\tau_2}\sigma_1 + \right.$$
$$\left. + y_2(x,t)e^{\tau_2+\tau_3}\sigma_2 + y_3(x,t)e^{\tau_3+\tau_2}\sigma_1 + y_3(x,t)e^{2\tau_3}\sigma_2\right] \quad \Rightarrow$$
$$\Rightarrow \widehat{M}(x,t,\tau) = \frac{g(x)}{2}\left[y_1(x,t)e^{\tau_1+\tau_2}\sigma_1 + y_1(x,t)e^{\tau_1+\tau_3}\sigma_2 + \right.$$
$$\left. + y_2(x,t)e^{2\tau_2}\sigma_1 + y_2(x,t)\sigma_2 + y_3(x,t)\sigma_1 + y_3(x,t)e^{2\tau_3}\sigma_2\right].$$

(note that in $\widehat{M}(x,t,\tau)$ there are no members containing $e^{\tau_1}$ measurement exponents $|m| = 1$);

$$S(x,t,\tau) \equiv \frac{g(x)}{2}\left(e^{\tau_2}\sigma_1 + e^{\tau_3}\sigma_2\right)\left[\sum_{2\le|m|\le N_z}^{*} y^m(x,t)e^{(m,\tau)} + \sum_{1\le|m|\le N_z}^{*} y^{e_1+m}(x,t)e^{(e_1+m,\tau)}\right] =$$

$$= \frac{g(x)}{2}\left[\sum_{2\le|m|\le N_z}^{*} y^m(x,t)\left(e^{\tau_2+(m,\tau)}\sigma_1+e^{\tau_3+(m,\tau)}\sigma_2\right)+\right.$$

$$\left.+\sum_{1\le|m|\le N_z}^{*} y^{e_1+m}(x,t)\left(e^{(e_1+m,\tau)+\tau_2}\sigma_1+e^{(e_1+m,\tau)+\tau_3}\sigma_2\right)\right]\ \Rightarrow$$

$$\Rightarrow \widehat{S}(x,t,\tau)=\frac{g(x)}{2}\left[\sum_{\substack{2\le|m|\le N_z,\\ m_2+1=m_3}} y^m(x,t)\sigma_1+\sum_{\substack{2\le|m|\le N_z,\\ m_3+1=m_2}} z^m(x,t)\sigma_2+\sum_{\substack{2\le|m|\le N_z,\\ m_2+1\ne m_3, m_3+1\ne m_2}}^{*} y^m(x,t)e^{(m,\tau)}+\right.$$

$$+\left(\sum_{\substack{1\le|m|\le N_z,\\ m_2+1=m_3}} y^{e_1+m}(x,t)\sigma_1+\sum_{\substack{1\le|m|\le N_z,\\ m_3+1=m_2}} y^{e_1+m}(x,t)\sigma_2\right)e^{\tau_1}+\sum_{\substack{1\le|m|\le N_z,\\ m_2+1\ne m_3, m_3+1\ne m_2}}^{*} y^{e_1+m}(x,t)e^{(e_1+m,\tau)}.$$

After embedding, the right-hand side of equation (16) will look like

$$\widehat{G}(x,t,\tau)=-\frac{\partial}{\partial x}\left[y_0(x,t)+\sum_{i=1}^{3}y_i(x,t)e^{\tau_i}+\sum_{2\le|m|\le N_z}^{*} y^m(x,t)e^{(m,\tau)}+\sum_{1\le|m|\le N_z}^{*} y^{e_1+m}(x,t)e^{(e_1+m,\tau)}\right]+$$

$$+\widehat{M}(x,t,\tau)+\widehat{S}(x,t,\tau)+R_1y(x,t,\tau)+p(x,t,\tau),$$

moreover, in $\widehat{S}(x,t,\tau)$ the coefficient at $e^{\tau_1}$ do not depend on $y_1(x,t)$. As indicated in [5], the embedding $G(x,t,\tau)\to\widehat{G}(x,t,\tau)$ will not affect the accuracy of the construction of asymptotic solutions of problem (2), since $\widehat{Z}(x,t,\tau)|_{\tau=\psi(x)/\varepsilon}\equiv Z(x,t,\tau)|_{\tau=\psi(x)/\varepsilon}$.

We show that the problem (9) has the unique solution in the space $U$ if (16) is satisfied.

*Theorem 2. Let the conditions (i)-(iv) take place and the right-hand side*

$$h(x,t,\tau)=h_0(x,t)+\sum_{i=1}^{3}h_i(x,t)e^{\tau_i}+\sum_{2\le|m|\le N_z}^{*} h^m(x,t)e^{(m,\tau)}+\sum_{1\le|m|\le N_H}^{*} h^{e_1+m}(x,t)e^{(e_1+m,\tau)}\in U$$

*satisfy the condition (13). Then the problem (9) is uniquely solvable in the space U under the additional condition (16).*

*Proof.* To use the condition (16), we calculate the expression $-\frac{\partial y}{\partial x}+\frac{g(x)}{2}(e^{\tau_2}+e^{\tau_3})y+R_1y+p(x,t,\tau)$. Since

$$-\frac{\partial}{\partial x}\left[y_0(x,t)+\sum_{i=1}^{3}y_i(x,t)e^{\tau_i}+\sum_{2\le|m|\le N_z}^{*} y^m(x,t)e^{(m,\tau)}+\sum_{1\le|m|\le N_z}^{*} y^{e_1+m}(x,t)e^{(e_1+m,\tau)}\right]+$$

$$+\widehat{M}(x,t,\tau)+\widehat{S}(x,t,\tau)+\sum_{i=1}^{3}\left[\left(I_i^0\left(K(x,t,s)y_i(s,t)\right)\right)_{s=x}e^{\tau_i}-\left(I_i^0\left(K(x,t,s)y_i(s,t)\right)\right)_{s=x_0}\right]+$$

$$+\sum_{1\le|m|\le N_z}^{*}\left[\left(I_m^0\left(K(x,t,s)y^m(s,t)\right)\right)_{s=x}e^{(m,\tau)}-\left(I_m^0\left(K(x,t,s)y^m(s,t)\right)\right)_{s=x_0}\right]+$$

$$+\sum_{1\le|m|\le N_z}^{*}\left[\left(I_{e_1+m}^0\left(K(x,t,s)y^{e_1+m}(s,t)\right)\right)_{s=x}e^{(e_1+m,\tau)}-\left(I_{e_1+m}^0\left(K(x,t,s)y^{e_1+m}(s,t)\right)\right)_{s=x_0}\right]+$$

$$+p(x,t,\tau).$$

therefore(16) takes the form

$$\frac{\partial\left(\xi(x,t)\right)}{\partial x}+\frac{K(x,t,x)}{\lambda_1(x)}\xi(x,t)+p_1(x,t)e^{\tau_1}+p_0(x,t)=0.$$

Taking into account the initial condition (15), this equation has the unique solution

$$\xi(x,t) = e^{q(x,t)} \left[ \xi(x_0,t) + \int\limits_{x_0}^{x} p_i(s,t) e^{-q(s,t)} ds \right],$$

where $q(x,t) = \int\limits_{x_0}^{x} \lambda_1^{-1}(s) K(s,t,s) ds$. Hence, under the conditions of Theorem 2, the solution (14) in the space $U$ is uniquely determined.

Applying theorems 1 and 2 to iterative problems ( we construct the series $(8_k)$, with coefficients in the class $U$. Let $y_{\varepsilon N}(x,t) = \sum\limits_{k=0}^{N} \varepsilon^k y_k(x,t,\psi(x,\varepsilon))$ be the restriction of the $N$-th partial sum of this series at $\tau = \psi(x,\varepsilon)$. As well as in [2], it is not difficult to prove the following

*Theorem 3. Let the conditions (i)-(iv) be satisfied. Then for $\varepsilon \in (0, \varepsilon_0]$, where $\varepsilon_0 > 0$ is sufficiently small, the problem (1) has the unique solution $y(x,t,\varepsilon) \in [x_0, X] \times [0,T]$, and the estimate*

$$\|y(x,t,\varepsilon - y_{\varepsilon N}(x,t)\|_{C([x_0,X]\times[0,T])} \le C_N \varepsilon^{N+1} \quad (N = 0,1,2,\ldots),$$

*takes place, where the constant $C_N > 0$ does not depend on $\varepsilon \in (0, \varepsilon_0]$.*

*3 Construction of the solution of the first iteration problem in space U*

Theorem 1, we will try to find a solution to the first iteration problem $(8_0)$. Since the right side $h(x,t)$ of the equation $(8_0)$ satisfies condition (13), this system has (according to (14)) a solution in space $U$ in the form

$$y_0(x,t,\tau) = y_0^{(0)}(x,t) + \alpha_1^{(0)}(x,t) e^{\tau_1}, \tag{17}$$

where $y_0^{(0)}(x,t)$ is the solution of the integrated equation

$$y_0^{(0)}(x,t) = \int\limits_{x_0}^{x} \left(-a^{-1}(x) K(x,t,s)\right) y_0^{(0)}(s,t) ds - a^{-1}(x) h(x,t),$$

where $\alpha_1^{(0)}(x,\ t) \in C^\infty[x_0,\ X] \times [0,\ T]$ are arbitrary functions. Subjecting (17) to the initial condition $y_0(x_0,t,0) = y^0$, we will have

$$y_0^{(0)}(x_0,t) + \alpha_1^{(0)}(x_0,t) = y^0 \quad \Leftrightarrow \quad \alpha_1^{(0)}(x_0,t) = y^0 + a^{-1}(x_0) h(x_0,t).$$

To fully compute the functions $\alpha_1^{(0)}(x,t)$, we proceed to the next iteration problem $(8_1)$. Substituting into it the solution (14) of the equation $(8_0)$ we arrive at the following equation:

$$\begin{gathered} L y_1(x,t,\tau) = -\frac{d}{dx} y_0^{(0)}(x,t) - \frac{d}{dx}\left(\alpha_1^{(0)}(x,t)\right) e^{\tau_1} + \\ + \frac{g(x)}{2}\left(e^{\tau_2}\sigma_1 + e^{\tau_3}\sigma_2\right)\left(y_0^{(0)}(x,t) + \alpha_1^{(0)}(x,t) e^{\tau_1}\right) + \\ + \frac{K(x,t,x)\alpha_1^{(0)}(x,t)}{\lambda_1(x)} e^{\tau_1} - \frac{K(x,t,x_0)\,\alpha_1^{(0)}(x_0,t)}{\lambda_1(x_0)}, \end{gathered} \tag{18}$$

(here we used the expression $(5_1)$ for $R_1 y(x,t,\tau)$ and took into account that for $y(x,t,\tau) = y_0(x,t,\tau)$ only the terms with $e^{\tau_1}$ remain in the sum $(5_1)$). It is not difficult to see that the right side

$$\begin{gathered} H(x,t,\tau) = -\frac{d}{dx} y_0^{(0)}(x,t) - \frac{d}{dx}\left(\alpha_1^{(0)}(x,t)\right) e^{\tau_1} + \\ + \frac{g(x)}{2}\left(e^{\tau_2}\sigma_1 + e^{\tau_3}\sigma_2\right)\left(y_0^{(0)}(x,t) + \alpha_1^{(0)}(x,t) e^{\tau_1}\right) + \end{gathered}$$

$$+\frac{K(x,t,x)\alpha_1^{(0)}(x,t)}{\lambda_1(x)}e^{\tau_1}-\frac{K\left(x,t,x_0\right)\alpha_1^{(0)}(x_0,t)}{\lambda_1(x_0)},$$

of equation (18) belongs to space $U$. Equation (18) is solvable in this space $U$ if and only if condition (13) are satisfied, which in our case take the form

$$-\frac{d}{dx}\left(\alpha_1^{(0)}(x,t)\right)+\frac{K(x,t,x)}{\lambda_1(x)}\alpha_1^{(0)}(x,t)=0.$$

Attaching to this system the initial conditions $\alpha_1^{(0)}(x_0,t)=y^0+a^{-1}(x_0)h(x_0,t)$, we find uniquely functions

$$\alpha_1^{(0)}(x,t)=\alpha_1^{(0)}(x_0,t)exp\left\{\int\limits_{x_0}^{x}\left(\frac{K(s,t,s)}{\lambda_1(s)}\right)ds\right\},$$

therefore, we uniquely calculate the solution (17) of the problem ($9_0$) in the space $U$. Moreover, the main term of the asymptotic of the solution to problem (2) has the form

$$y_{\varepsilon 0}(x,t)=y_0^{(0)}(x,t)+\alpha_1^{(0)}(x_0,t)exp\left\{\int\limits_{x_0}^{x}\left(\frac{K(s,t,s)}{\lambda_1(s)}\right)ds\right\}e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{x}\lambda_1(\theta)d\theta},$$

where $\alpha_1^{(0)}(x_0,\ t)=y^0+a^{-1}(x_0)h(x_0,\ t)$, $y_0^{(0)}(x,\ t)$ is the solution of the integrated system $y_0^{(0)}(x,t)= =\int\limits_{x_0}^{x}\left(-a^{-1}(x)K(x,t,s)\right)y_0^{(0)}(s,t)ds-a^{-1}(x)h(x,t)$.

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## Жылдам осцилляцияланатын коэффициентті дербес туындылы скаляр интегро-дифференциалдық теңдеулердің асимптотикалық шешімдері

Мақалада микро-біртекті емес ортадағы әртүрлі физикалық процестерді сипаттайтын жылдам осцилляцияланатын коэффициентті дербес туындылы сингуляр ауытқыған интегро-дифференциалдық теңдеу үшін Коши есебін асимптотикалық интегралдау алгоритмін әзірлеуге арналған. Бұл дербес туындылы дифференциалдық теңдеулер теориясындағы бағыт қарқынды дамып келеді және радиофизикада, электротехникада, сүзу теориясында, фазалық өту теориясында, серпімділік теориясында және физиканың, механика мен техниканың басқа да бөлімдерінде көптеген қолданыс табады. Мұндай процестерді зерттеу үшін әдетте асимптотикалық әдістер қолданылады. Қазіргі уақытта қарқынды дамып келе жатқан сандық әдістер асимптотикалық әдістерді жоққа шығармайтыны белгілі. Бұл бірнеше себептерге байланысты болады. Біріншіден, негізделген асимптотикалық шешім, әсіресе оның негізгі мүшесі, шешімнің сапасы туралы қосымша ақпаратты алуға және осы мағынада көбінесе табуы мүмкін болмаған нақты шешімді ауыстыруға мүмкіндік береді. Екіншіден, жоғарыда айтылғандай, шешім құрылымын білу күрделі есептерді шешудің сандық әдістерін жасауға көмектеседі, сондықтан асимптотикалық әдістердің дамуы сандық әдістердің дамуына ықпал етеді. Есептің регуляризациясы жүргізілген, жалпы итерациялык есептердің қалыпты және бір мәнді шешілімділігі дәлелденген.

*Кілт сөздер:* сингуляр ауытқу, дербес туындылы дифференциалдық теңдеулер, интегралды регуляризациялау, итерациялық есептердің шешімділігі.

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## Асимптотические решения скалярных интегро-дифференциальных уравнений с частными производными и с быстро осциллирующими коэффициентами

Статья посвящена разработке алгоритма асимптотического интегрирования задачи Коши для сингулярно возмущенного интегро-дифференциального уравнения в частных производных с быстро осциллирующими коэффициентами, описывающими различные физические процессы в микронеоднородных средах. Это направление в теории уравнений с частными производными интенсивно развивается и находит многочисленные применения в радиофизике, электротехнике, теории фильтрации, теории фазовых переходов, теории упругости и других разделах физики, механики и техники. Для исследования таких процессов обычно используются асимптотические методы. Известно, что бурно развивающиеся в настоящее время численные методы не исключают асимптотических. Это происходит по ряду причин. Во-первых, разумно построенная асимптотика, особенно ее главный член, несет существенную для приложений информацию о качественном поведении решения и в этом смысле в определенной мере заменяет точное решение, которое чаще всего не может быть найдено. Во-вторых, как это следует из сказанного выше, знание структуры решения помогает при разработке численных методов решения сложных задач, поэтому развитие асимптотических методов способствует развитию численных методов. Произведена регуляризация задачи, доказана нормальная и однозначная разрешимость общих итерационных задач.

*Ключевые слова:* сингулярное возмущение, дифференциальное уравнение с частными производными, регуляризация интеграла, разрешимость итерационных задач.

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# Constructing the fundamental solution to a problem of heat conduction

In this article, we discuss auxiliary initial-boundary value problems which will subsequently be used to solve boundary-value problem of heat conduction with axial symmetry in a degenerating domain. One of the problems is posed with homogeneous boundary conditions in order to construct a fundamental solution that is used to determine thermal potentials. The initial condition contains the Dirac function. The solution to the problems is found explicitly using the Laplace integral transformation. The boundary value problem is also considered in the absence of axial symmetry. It is shown that this problem splits into families of boundary-value problems similar to the problems considered above. In conclusion, we state the boundary value problem of heat conduction with axial symmetry in a degenerating domain, and its fundamental solution, found above, is written out.

*Keywords:* equation of heat conduction, fundamental solution, Laplace transformation, axial symmetry, Bessel equation.

## *Introduction*

*Problem I.* In the domain $\Omega_\infty = \{(r,\, t) \,:\quad 0 < r < \infty;\ t > 0\}$ we consider the boundary value problem for equation

$$\frac{\partial u}{\partial t} = \frac{a^2}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right), \tag{1}$$

satisfying the boundary condition

$$\lim_{r\to 0+}\frac{u(r,\, t)}{\ln r} = -\varphi(t),\ t > 0, \tag{2}$$

$$\lim_{r\to\,+\infty} u(r,\, t) = 0, \tag{3}$$

*Problem II.* In the domain $\Omega_\infty = \{(r,\, t) \,:\ 0 < r < \infty;\ t > 0\}$ we consider the boundary value problem for equation (1) under boundary conditions (2), (3) and and initial condition

$$u(r,\, 0) = 0,\ r > 0. \tag{4}$$

It is known that equation (1) follows from the equation

$$\frac{\partial u}{\partial t} = a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right),$$

passing to polar coordinates.

*Problem III.* In a case without axial symmetry, we consider the following problem in the domain

$$\Omega_1 = \{(r;\ \alpha;\ t) \,:\, 0 < r < t;\ 0 \le \alpha \le 2\pi;\ 0 < t < T\}$$

find the solution to the equation

$$\frac{\partial u}{\partial t} = a^2\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\cdot\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \alpha^2}\right],$$

satisfying the boundary conditions

$$\lim_{r\to 0+}\frac{u(r,\,\alpha,\,t)}{\ln{(1/r)}}=u_0(t);\;0<t<T,$$

$$\lim_{r\to t-}u(r,\,\alpha,\,t)=u_1(\alpha;\;t)\equiv u_c(x;\;y;\;t)|_{\sqrt{x^2+y^2}=t};\;(\alpha;\;t)\in\partial\Omega_1,$$

where $\partial\Omega_1$ is the lateral surface of the cone.

Earlier [1–4] we studied a homogeneous problem for the heat equation in the angular domain $G=\{(x;t):\;t>0, 0<x<t\}$ (as the domain $\Omega$):

find a solution to the heat equation

$$\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2},$$

satisfying the boundary conditions:

$$u(x,\;t)|_{x=0}=0,\quad u(x,\;t)|_{x=t}=0.$$

In the work [5] in the domain $G$ the boundary-value problem of a homogeneous heat equation with boundary conditions:

$$\left.\frac{\partial u}{\partial x}\right|_{x=0}=0,\left.\frac{\partial u}{\partial x}\right|_{x=t}=0.$$

was considered.

Solving the boundary value problems was reduced to solving the Volterra integral equation of the second kind with a kernel

$$K(t,\,\tau)=\frac{1}{2a\sqrt{\pi}}\left\{\frac{t+\tau}{(t-\tau)^{\frac{3}{2}}}\exp\left(-\frac{(t+\tau)^2}{4a^2(t-\tau)}\right)+\frac{1}{(t-\tau)^{\frac{1}{2}}}\exp\left(-\frac{t-\tau}{4a^2}\right)\right\}.$$

It is shown that the kernels of the obtained integral equations are “incompressible”, that is, the norm of the integral operator acting in the class of continuous functions is equal to unity. By the Carleman-Vekua method, solving the integral equation was reduced to solving the nonhomogeneous Abel equation. The explicit form of the solution of the integral equation has allowed to estimate the solution to the posed boundary value problem and precisely to determine the uniqueness classes of the solution to the posed homogeneous problem.

In [6], along with the direct problem, the conjugate boundary-value problem for the heat equation in the weighted functional class was also studied, and it was established that the posed boundary value problem is Noetherian problem.

We also note that boundary value problems for a spectrally loaded parabolic equation reduce to this kind of singular integral equations, when the load line moves according to the law $x=t$ [7–11] and problems for essentially loaded equation of heat conduction [12].

In all works, the boundary of the domain moves at a constant velocity. Attempts to study the solvability of boundary value problems for the heat equation in non-cylindrical domains with a variable velocity of changing the boundary were made in works [13–14].

In works [15–17] the second-order Volterra singular integral equation with the above kernel $K(t,\,\tau)$ is investigated. The multiplicity of eigenvalues and eigenfunctions for the Volterra integral operator is determined depending on the value of the spectral parameter and its spectrum is found.

In this paper, assuming that the isotropy property is fulfilled in the angular coordinate (axial symmetry), we study the problem for the heat equation in polar coordinates, to which the two-dimensional problem in the spatial variable is reduced.

In [18], the two-dimensional Dirichlet problem for the heat equation with respect to the spatial variable in an infinite dihedral angle was also considered. Using the Fourier transformation, the problem was reduced to a one-dimensional boundary value problem with the parameter.

Now we are studying the boundary value problem for the heat equation in the cone. To construct a solution to the problem we consider two auxiliary problems I and II.

The problem I solved in paragraph 1 is necessary to construct a fundamental solution, which will be further used in determining the thermal potentials. The solution to the original problem will be further presented as a sum of thermal potentials.

The solution $u_1(r,t)$ to problem II found in paragraph 2 is used in the integral representation of the original problem to annul the boundary condition at the boundary $x = 0$. In paragraph 3, we have formulated a result that follows from the contents of paragraphs 1 and 2.

In paragraph 4, a boundary value problem is considered in the absence of axial symmetry that is problem III. It is shown that this problem splits into families of boundary-value problems similar to the problem considered in paragraph 1.

*1 Function of the thermal instantaneous point source*

We will seek its solution in the class of originals of the Laplace transformation with respect to the variable $t$, depending on the parameter $r,\ r > 0$.

We introduce the notation for the Laplace image: $L[u(r,\, t)] = \overline{u}(r,\, p)$.

As a result of applying the transformation to the equation (1):

$$u_t = \frac{a^2}{r}\,(u_r + r\,u_{rr}) = \frac{a^2}{r}\,u_r + a^2 u_{rr},$$

taking into account the property of the Laplace transformation:

$$u_t \div p\,\overline{u}(r,\, p) - u(r,\, 0),$$

and to conditions (2) and (3), we obtain in the domain $\{r,\ r > 0\}$ the boundary-value problem for the ordinary differential equation:

$$\frac{d^2\overline{u}}{dr^2} + \frac{1}{r}\,\frac{d\overline{u}}{dr} - \frac{p}{a^2}\overline{u} = -\frac{\delta(r - r_0)}{a^2 r}, \tag{5}$$

$$\lim_{r\to 0+} \frac{\overline{u}(r,\, p)}{\ln r} = -\overline{\varphi}(p), \tag{6}$$

$$\lim_{r\to +\infty} \overline{u}(r,\, p) = 0. \tag{7}$$

The homogeneous equation corresponding to equation (5) as a result of the replacement: $z = \frac{\sqrt{p}}{a} r$ is transformed to a modified Bessel equation:

$$\frac{d^2\,\overline{u}}{dz^2} + \frac{1}{z}\,\frac{d\overline{u}}{dz} - \overline{u} = 0. \tag{8}$$

The solution of equation (8) has the form: ([1], formula 8.494(1))

$$\overline{u_{hom}}(z) = C_1\, I_0(z) + C_2\, K_0(z), \tag{9}$$

where ([19], formula 8.447):

$$I_0(z) = \sum_{n=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2n}}{(n!)^2};\ K_0(z) = -\ln z I_0(z) + \sum_{n=0}^{\infty} \frac{z^{2n}}{2^{2n}\,(n\,!)^2}\,\psi(n+1);$$

$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$ is Euler psi-function.

The following asymptotic formulas and approximations hold [19]:

when $0 < z << 1$

$I_0(z) \approx 1;\ K_0(z) \approx \ln\frac{2}{Cz},\ \ C \approx 0,57721...$– Euler *const*.

when $z >> 1$

$$I_0(z) = \frac{e^z}{\sqrt{2\pi\, z}}\left\{1 + \frac{1}{8z} + \frac{9}{128z^2} + \left(\frac{1}{z^3}\right)\right\},$$

$$K_0(z) = \sqrt{\frac{\pi}{2z}}\,e^{-z}\left[\sum_{k=0}^{n-1} \frac{1}{(2z)^k}\ \cdot \frac{\Gamma\left(k+\frac{1}{2}\right)}{k\,!\,\Gamma\left(-k+\frac{1}{2}\right)} + \Theta_3\frac{\Gamma\left(n+\frac{1}{2}\right)}{(2z)^n\ n!\ \left(-n+\frac{1}{2}\right)}\right].$$

Here:

$|\Theta_3| < 1$ and $Re\Theta_3 \geq 0$, when $Rez \geq 0$;

$|\Theta_3| < |co\,\sec(\arg z)|$, when $Imz = 0$;
$0 \le |\Theta_3| \le 1$, when $Rez \ge 0$.

Returning to the variable $r$ in (9), we obtain the general solution of the homogeneous equation corresponding to the equation (5):

$$\overline{u_{hom}}(r,\,p) = C_1 I_0\left(\frac{\sqrt{p}}{a}\,r\right) + C_2 K_0\left(\frac{\sqrt{p}}{a}\,r\right).$$

Then, according to the method of variation of arbitrary constants, we write the general solution of equation (5) in the form:

$$\overline{u}(r,\,p) = C_1(r) I_0\left(\frac{r\sqrt{p}}{a}\right) + C_2(r) K_0\left(\frac{r\sqrt{p}}{a}\right). \tag{10}$$

To determine the functions $C_1(r)$, $C_2(r)$ we compose a system of equations:

$$\begin{cases} C_1^{/}(r)\, I_0\left(\frac{r\sqrt{p}}{a}\right) + C_2^{/}(r)\, K_0\left(\frac{r\sqrt{p}}{a}\right) = 0, \\ C_1^{/}(r)\, \frac{d}{dr}\left(I_0\left(\frac{r\sqrt{p}}{a}\right)\right) + C_2^{/}(r) \frac{d}{dr}\left(K_0\left(\frac{r\sqrt{p}}{a}\right)\right) = \frac{\delta(r-r_0)}{r}. \end{cases}$$

After some simplifications, taking into account formula 8.486 from [19], we obtain:

$$\begin{cases} C_1^{/}(r)\, I_0\left(\frac{r\sqrt{p}}{a}\right) + C_2^{/}(r)\, K_0\left(\frac{r\sqrt{p}}{a}\right) = 0, \\ -C_1^{/}(r)\, I_1\left(\frac{r\sqrt{p}}{a}\right) + C_2^{/}(r) K_1\left(\frac{r\sqrt{p}}{a}\right) = \frac{\delta(r-r_0)}{a\,r\sqrt{p}}. \end{cases}$$

By virtue of formula 8.477 (2) from [19], the determinant of this system is equal to:

$$\Delta = \,I_0\left(\frac{r\sqrt{p}}{a}\right) K_1\left(\frac{r\sqrt{p}}{a}\right) + \,I_1\left(\frac{r\sqrt{p}}{a}\right) K_0\left(\frac{r\sqrt{p}}{a}\right) = \frac{a}{r\sqrt{p}}.$$

Then the solution of the system takes the form:

$$C_1^{/}\left(r\right) = -\frac{\delta\left(r - r_0\right)}{a^2} K_0\left(\frac{r\sqrt{p}}{a}\right);$$

$$C_2^{/}\left(r\right) = \frac{\delta\left(r - r_0\right)}{a^2} I_0\left(\frac{r\sqrt{p}}{a}\right).$$

Integrating the last equalities, we obtain

$$C_1\left(r\right) = -\int_r^{+\infty} \frac{\delta\left(\xi - r_0\right)}{a^2} K_0\left(\frac{\xi\sqrt{p}}{a}\right) d\xi =$$

$$= C_1^0 + \begin{cases} \frac{1}{a^2} K_0\left(\frac{r_0\sqrt{p}}{a}\right); & r < r_0, \\ 0; \quad r > r_0. \end{cases}$$

and

$$C_2\left(r\right) = \int_0^r \frac{\delta\left(\xi - r_0\right)}{a^2} I_0\left(\frac{\xi\sqrt{p}}{a}\right) d\xi \; =$$

$$= C_2^0 + \begin{cases} 0; \quad r < r_0. \\ \frac{1}{a^2} I_0\left(\frac{r_0\sqrt{p}}{a}\right); & r > r_0, \end{cases}$$

We substitute the found functions $C_1\left(r\right),\, C_2\left(r\right)$ into (10):

$$\bar{u}(r,\,p) = C_1^0 I_0\left(\frac{r\,\sqrt{p}}{a}\right) + C_2^0 K_0\left(\frac{r\,\sqrt{p}}{a}\right) + \bar{G}(r,\,r_0,\,p),$$

where

$$\bar{G}(r,\,r_0,\,p) = \begin{cases} \frac{1}{a^2} K_0\left(\frac{r\sqrt{p}}{a}\right) I_0\left(\frac{r\sqrt{p}}{a}\right); \; 0 < r < r_0 \\ \frac{1}{a^2} I_0\left(\frac{r\sqrt{p}}{a}\right) K_0\left(\frac{r\sqrt{p}}{a}\right); \; r_0 < r < \infty \end{cases}$$

Now we will define the values of the constants. Let be $\forall p: \; Rep > 0$.

Then, by virtue of asymptotic formulas and approximations, we have:

$$r \to +\infty \Rightarrow \begin{cases} I_0\left(\frac{\sqrt{p}\,r}{a}\right) \to +\infty, \\ K_0\left(\frac{\sqrt{p}\,r}{a}\right) \to 0. \end{cases}$$

Therefore, to satisfy condition (7), it is necessary to set $C_1 = 0$.

When $r \to 0$ (at $\forall p: \; Rep > 0$) we get

$$\frac{I_0\left(\frac{r\sqrt{p}}{a}\right)}{\ln r} \to 0, \qquad \frac{K_0\left(\frac{r\sqrt{p}}{a}\right)}{\ln r} \to -1$$

From condition (6) we have

$$\lim_{r\to 0}\frac{\bar{u}\,(r,\,p)}{\ln\,r} = \lim_{r\to 0+}\frac{C_2}{\ln\,r}\ln\left(\frac{2a}{C\sqrt{p}\,r}\right) = -C_2 = -\bar{\varphi}(p).$$

We have obtained a solution to problem (5) – (7)

$$\bar{u}\,(r,\,p) = \bar{\varphi}(p)\,K_0\left(\frac{r\,\sqrt{p}}{a}\right) + \bar{G}(r,\,r_0,\,p). \tag{11}$$

By virtue of the formula [20], (p.241; formula No.117)

$$L\left[\frac{1}{2t}\cdot e^{-\frac{r^2}{4a^2\,t}}\right] = K_0\left(\frac{r\,\sqrt{p}}{a}\right)$$

and

$$L\left[\frac{1}{2t}\exp\left(-\frac{a+b}{2t}\right)\,I_0\left(\frac{a-b}{2t}\right)\right] = K_0\left(\sqrt{ap}+\sqrt{bp}\right)\,I_0\left(\sqrt{ap}-\sqrt{bp}\right)$$

when $Rea > Reb > 0$, after applying inverse Laplace transform to (11) and some simplifications, we obtain:

$$u(r,\,t) = u_1(r,\,t) + \frac{1}{2a^2t}\exp\left(-\frac{r^2+r_0^2}{4a^2t}\right)\cdot I_0\left(\frac{r\,r_0}{2a^2t}\right), \tag{12}$$

where

$$u_1(r,\,t) = \left(\frac{1}{2t}\exp\left(-\frac{r^2}{4a^2t}\right)\right) * \varphi(t) = \int_0^t \frac{\exp\left(-\frac{r^2}{4a^2(t-\tau)}\right)}{2(t-\tau)}\,\varphi\,(\tau)\;d\tau. \tag{13}$$

(12) is the solution to problem (1)–(3) and the initial condition

$$u_1(r,\,0) = \frac{\delta(r-r_0)}{r};\;\, 0 < r < \infty,\;\, 0 < r_0 < +\infty,$$

which is verified directly. For example, after replacement $z = \frac{r}{2a\,\sqrt{t-\tau}}$ function (13) takes the form

$$u_1(r,\,t) = \int_{\frac{r}{2a\sqrt{t}}}^{\infty}\frac{1}{z}\,e^{-z^2}\cdot\varphi\left(t-\frac{r^2}{4a^2\,z^2}\right)dz$$

Then condition (2) can be written as

$$\lim_{r\to 0}\varphi(t)\cdot\frac{1}{\ln r}\int_{\frac{r}{2a\sqrt{t}}}^{\infty}\frac{e^{-z^2}}{z}\,dz \;=\lim_{r\to 0}\frac{\varphi(t)}{2\ln r}Ei\left(-\left(\frac{r}{2a\sqrt{t}}\right)^2\right) = \varphi(t),$$

because ([19], 8.214 (1)) from the representation

$$Ei\;(x)\; = C + \ln(-x) + \sum_{k=0}^{\infty}\frac{x^k}{k\cdot k!}$$

we have

$$\lim_{r\to 0}\frac{Ei\left(-\left(\frac{r}{2a\sqrt{t}}\right)^2\right)}{2\ln r} = \lim_{r\to 0}\frac{\ln\frac{r}{2a\sqrt{t}}}{\ln r} = 1.$$

*2 The first boundary value problem for a semirestricted domain*

In the domain $\Omega_\infty = \{(r,\, t)\,:\ 0<r<\infty;\ t>0\}$ we consider the boundary value problem for equation (1) under boundary conditions (2)–(3). This problem occurs in the theory of a diffusion trace behind a drop and a solid particle.

We will seek its solution in the class of originals of the Laplace transform with respect to the variable $t$, depending on the parameter $r$, $r>0$. In this paragraph we use the results of paragraph 2.

As a result, we get

$$u_1(r,t) = \left(\frac{1}{2t}\exp\left(-\frac{r^2}{4a^2t}\right)\right) * \varphi(t) = \int_0^t \frac{\exp\left(-\frac{r^2}{4a^2(t-\tau)}\right)}{2(t-\tau)}\varphi(\tau)d\tau \tag{14}$$

(14) is the solution to problem (1)–(4) and this fact is verified directly.

*3 Main result*

From the contents of paragraphs 1 and 2, the following theorem is proved.

*Theorem.* The function

$$u(r,\, t) = \int_0^t \frac{\exp\left(-\frac{r^2}{4a^2(t-\tau)}\right)}{2(t-\tau)}\varphi(\tau)d\tau + \frac{1}{2a^2t}\exp\left(-\frac{r^2+r_0^2}{4a^2t}\right)\cdot I_0\left(\frac{r\,r_0}{2a^2t}\right),$$

where $\varphi(t)$ is a continuous function for $t\in(0,+\infty)$ and $|\varphi(t)|\le Mt^{-1}$ , *const* $M>0$, is the solution to the problem (1)–(4).

*4 Case without axial symmetry*

In the domain

$$\Omega_1 = \{(r;\ \alpha;\ t)\,:\,0<r<t;\ 0\le\alpha\le 2\pi;\ 0<t<T\}$$

find the solution to the equation

$$\frac{\partial u}{\partial t} = a^2\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\cdot\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial\alpha^2}\right], \tag{15}$$

satisfying the boundary conditions

$$\lim_{r\to 0+}\frac{u(r,\,\alpha,\,t)}{\ln(1/r)} = u_0(t);\ 0<t<T, \tag{16}$$

$$\lim_{r\to t-} u(r,\,\alpha,\,t) = u_1(\alpha;\ t) \equiv u_c(x;\ y;\ t)|_{\sqrt{x^2+y^2}=t};\ (\alpha;\ t)\in\partial\Omega_1, \tag{17}$$

where $\partial\Omega_1$ is the lateral surface of the cone.

To the boundary problem (15)–(17) we apply the Fourier method (the method of separation of variables). We seek the desired solution $u(r,\alpha,t)$ in the form

$$u(r,\alpha,t) = U(r,t)\theta(\alpha) \tag{18}$$

Substituting (18) into (15) we get

$$\theta(\alpha)\cdot U_t = a^2\left[\frac{1}{r}\,(r\cdot U_r)_r^{/}\cdot\theta(\alpha) + \frac{1}{r^2}U\cdot\theta^{//}(\alpha)\right]$$

or

$$\frac{r^2}{a^2}\cdot\frac{U_t - \frac{a^2}{r}\,(r\cdot U_r)_r^{/}}{U} = \frac{\theta^{//}(\alpha)}{\theta(\alpha)} = -\lambda,$$

where $\lambda$ is a non-negative *const*.

We get the system of differential equations

$$\begin{cases} \theta^{//}(\alpha) + \lambda\theta(\alpha) = 0, \\ U_t - \frac{a^2}{r}\left(r \cdot U_r\right)_r^{/} + \frac{a^2\lambda}{r^2}U = 0. \end{cases} \tag{19}$$

The solution to the spectral problem

$$\begin{cases} \theta^{//}(\alpha) + \lambda\theta(\alpha) = 0, \\ \theta(0) = \theta(2\pi) \end{cases}$$

is a system of orthonormal eigenfunctions and eigenvalues

$$\theta_n(\alpha) = \frac{1}{\sqrt{2\pi}}\exp(in\alpha);\ \lambda_n = n^2;\ n \in Z. \tag{20}$$

The solution to problem (15)–(17) has the form

$$u(r,\,\alpha,\,t) = \sum_{n\in Z} U_n(r,\,t)\cdot\theta_n(\alpha). \tag{21}$$

When $\lambda_n = n^2$ for the second equation of the system (19) we obtain

$$\frac{\partial U_n}{\partial t} - \frac{a^2}{r}\frac{\partial}{\partial r}\left(r\cdot\frac{\partial U_n}{\partial r}\right) + \frac{a^2n^2}{r^2}U_n = 0 \tag{22}$$

For the function (21) we apply the condition (16):

$$\lim_{r\to 0}\left[\frac{U_0(r,\ t)}{\ln(1/r)} + \sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{U_n(r,\,t)\cdot\theta_n(\alpha)}{\ln(1/r)}\right] = u_0(t).$$

Expanding the function $u_0(t)$ in a Fourier series on the eigenfunctions $\theta_n(\alpha)$, we obtain

$$u_0(t) = \sum_{n\in Z} c_n(t)\cdot\theta_n(\alpha),$$

where

$$c_n(t) = \int_0^{2\pi} u_0(t)\cdot\theta_n(\alpha)d\alpha,$$

From here we get the condition for the unknown functions $U_n(r,t), \quad n\in\mathbb{Z}$:

$$\lim_{r\to 0}\frac{U_0(r,\ t)}{\ln(1/r)} = u_0(t); \tag{23}$$

$$\lim_{r\to 0}\frac{U_n(r,\ t)}{\ln(1/r)} = 0, \qquad n\in\mathbb{Z}\setminus\{0\} \tag{24}$$

For the function (21) we apply the condition (17):

$$\lim_{r\to t} u(r,\,\alpha,\,t) = \lim_{r\to t}\sum_{n\in Z} U_n(r,t)\cdot\theta_n(\alpha) = u_1(\alpha,\,t).$$

Expanding the function $u_1(\alpha,\,t)$ in a Fourier series on the eigenfunctions $\theta_n(\alpha)$, we obtain

$$u_1(\alpha,t) = \sum_{n\in Z} u_{1n}(t)\cdot\theta_n(\alpha),$$

where

$$u_{1n}(t) = \int_0^{2\pi} u_1(\alpha,t)\cdot\theta_n(\alpha)d\alpha. \tag{25}$$

Then from the equality

$$\sum_{n\in Z}\left(\lim_{r\to t}U_n(r,\,t)\right)\cdot\theta_n(\alpha)=\sum_{n\in Z}u_{1n}(t)\cdot\theta_n(\alpha)$$

we get one more condition for the unknown functions $U_n(r,t),\quad n\in\mathbb{Z}$

$$\lim_{r\to t}U_n(r,t)=u_{1n}(t),\tag{26}$$

where the functions $u_{1n}(t)$ are defined by equality (25).

We introduce the replacement of an unknown function $U_n(r,t),\quad n\in\mathbb{Z}$ by the formula

$$U_n(r,t)=v_n(r,t)\exp\left\{-\frac{a^2n^2}{r^2}\,t\right\}.\tag{27}$$

Then substituting replacement (27) into equation (22) and into conditions (23), (24) and (26), we obtain boundary value problems for determining a new unknown function $v_n(r,t)$:

*I*. $n=0$.

$$\frac{\partial v_0}{\partial t}=\frac{a^2}{r}\,\frac{\partial}{\partial r}\,\left(r\cdot\frac{\partial v_0}{\partial r}\right),\tag{28}$$

$$\lim_{r\to 0}\frac{v_0(r,\,t)}{\ln\left(1/r\right)}=u_0(t),\tag{29}$$

$$\lim_{r\to t}v_0(r,\,t)=u_{10}(t),\tag{30}$$

where

$$u_{10}(t)=\int_0^{2\pi}u_1(\alpha,t)d\alpha.$$

*II*. $n\neq 0$.

$$\frac{\partial v_n}{\partial t}=\frac{a^2}{r}\,\frac{\partial}{\partial r}\,\left(r\cdot\frac{\partial v_n}{\partial r}\right),\tag{31}$$

$$\lim_{r\to 0}\frac{v_n(r,\,t)}{\ln\left(1/r\right)}=0,\tag{32}$$

$$\lim_{r\to t}v_n(r,\,t)=u_{1n}(t)\tag{33}$$

Thus, we have obtained a family of boundary value problems (28)–(30) and (31)–(33), each of which is a boundary problem of the form (1)–(3). The issues of solvability of these boundary-value problems will be investigated later.

Solving boundary value problems (28)–(30) and (31)–(33), we find functions $\{v_n(r,\,t),\ n\in\mathbb{Z}\}$, and further, using (20)–(21) and replacement (27), we formally construct a series

$$u(r,\alpha,t)=\sum_{n\in\mathbb{Z}}v_n(r,t)\exp\left\{-\frac{a^2n^2}{r^2}t+in\alpha\right\}.$$

It is known that the series (formula 5.4.11.2 from [22], p.585):

$$\sum_{n\in\mathbb{Z}}\exp\left\{-\frac{a^2n^2}{r^2}t+i\,n\,\alpha\right\},$$

converges for $\forall\,t>0$.

*Remark.* The justification of the passage to the limit under the sign of the sum in all the series below follows from the uniform convergence of these series [21].

*Conclusion*

In the second part of the research, we solve the problem in the domain of degenerating to a point at the initial moment of time:

In the domain $\Omega = \{(r,\, t)\ :\ \ 0 < r < t;\ t > 0\}$ to find a solution to equation (1)

$$\frac{\partial u}{\partial t} = \frac{a^2}{r}\,\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right),$$

satisfying the boundary conditions (2)

$$\lim_{r\to 0+}\frac{u(r,\,t)}{\ln r} = -\varphi(t),\ \ t > 0,$$

and

$$\lim_{r\to t-} u(r,\,t) = -\psi(t),\ \ t > 0.$$

The function $G(r,\xi,t) = \xi G_0(r,\xi,t)$, where

$$G_0(r,\xi,t) = \frac{1}{2a^2 t}\exp\left\{-\frac{r^2+\xi^2}{4a^2 t}\right\}\cdot I_0\left(\frac{r\xi}{2a^2 t}\right),$$

is a fundamental solution to equation (1), $\xi$ is parameter. We note that this function was defined in Theorem. Thermal potentials will be preliminarily constructed using this fundamental solution.

Further, on the basis of the integral representation of the solution of the boundary value problem in the form of a sum of thermal potentials, we will reduce the study of the original problem to the study of the Volterra integral equation of the second kind, following [21] and [1–6].

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# Бір жылуөткізгіштік есебінің фундаментальды шешімін құру

Мақалада осьтік симметриялы жойылатын облыстағы жылуөткізгіштіктің шеттік есептерін шығару үшін қолдануға болатын көмекші бастапқы-шеттік есептер талқыланған. Біртекті шекаралық шарттарымен қойылған есептердің біреуі жылу потенциалдарын анықтау үшін қолданатын фундаментальды шешімді құру үшін қойылған. Бастапқы шарты Дирак функциясын қамтиды. Есептің шешімі Лаплас интегралдық түрлендіруі көмегімен айқын түрде табылған. Сонымен қатар, осьтік симметрия болмаған жағдайдағы шеттік есеп қарастырылды. Бұл есеп жоғарыда қарастырылған ұқсас шекаралық есептердің шоғырына бөлінетіні көрсетілген. Қорытынды бөлімінде осьтік симметриялы жойылатын облыстағы жылуөткізгіштік шеттік есебінің қойылуы көрсетілген және оның жоғарыда табылған фундаментальды шешімі жазылған.

*Кілт сөздер:* жылуөткізгіштік теңдеу, фундаментальды шешім, Лаплас түрлендіруі, осьтік симметрия, Бессель теңдеуі.

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# Построение фундаментального решения одной задачи теплопроводности

В статье обсуждены вспомогательные начально-краевые задачи, которые впоследствии будут использованы для решения краевой задачи теплопроводности с осевой симметрией в вырождающейся области. Одна из задач с однородными граничными условиями поставлена для построения фундаментального решения, которое используется для определения тепловых потенциалов. Начальное условие содержит функцию Дирака. Решение задач найдено в явном виде с помощью интегрального преобразования Лапласа. Также рассмотрена краевая задача при отсутствии осевой симметрии. Показано, что эта задача разбивается на семейства краевых задач, аналогичных рассмотренным выше. В заключении приведена постановка краевой задачи теплопроводности с осевой симметрией в вырождающейся области и выписано ее фундаментальное решение, найденное выше.

*Ключевые слова:* уравнение теплопроводности, фундаментальное решение, преобразование Лапласа, осевая симметрия, уравнение Бесселя.

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# Spectral problem for the sixth order nonclassical differential equations

In this article we investigate the correctness of boundary value problems for a sixth order quasi-hyperbolic equation in the Sobolev space

$$Lu = -D_t^6 u + \Delta u - \lambda u$$

($D_t = \frac{\partial}{\partial t}$, $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ – Laplace operator, $\lambda$ – real parameter). For the given operator $L$ two spectral problems are introduced and uniqueness of these problems is established. The eigenvalues and eigenfunctions of the first spectral problem are calculated for the sixth order quasi-hyperbolic equation. In this work we show that the equation $Lu = 0$ for $\lambda < 0$ under uniform conditions has a countable set of nontrivial solutions. Usually, this does not happen when the operator $L$ is an ordinary hyperbolic operator.

*Keywords:* a sixth order quasi-hyperbolic equation, eigenvalues, eigenfunctions, nontrivial solutions.

*Formulation of the problem*

Let $\Omega$ – be the limited area of space $\mathbb{R}^n$ variables $x_1, x_2, ..., x_n$ with smooth compact boundary $\Gamma = \partial\Omega$. Let's consider the following differential operator in the cylindrical area $Q = \Omega \times (0,T)$, $S = \Gamma \times (0,T)$, $0 < T < +\infty$

$$Lu \equiv -\frac{\partial^6 u}{\partial t^6} + \Delta u - \lambda u = f(x,t), \quad x \in \Omega, \quad t \in (0,T), \tag{1}$$

where $f(x,t)$ is a given function.

*Boundary value problem $I_{3,\lambda}$:* It is required to find a function $u(x,t)$ which is a solution to equation (1) in the cylinder $Q$ that satisfies following conditions

$$u(x,t)|_S = 0, \tag{2}$$

$$u(x,0) = \frac{\partial u}{\partial t}(x,0) = \frac{\partial^2 u}{\partial t^2}(x,0) = \frac{\partial^3 u}{\partial t^3}(x,0) = 0, \ x \in \Omega, \tag{3}$$

$$\frac{\partial u}{\partial t}(x,T) = \frac{\partial^2 u}{\partial t^2}(x,T) = 0, \ x \in \Omega. \tag{4}$$

*Boundary value problem $II_{3,\lambda}$:* It is required to find a function $u(x,t)$ which is a solution to equation (1) in the cylinder $Q$ that satisfies conditions (2), (3) and

$$D_t^4 u(x,t)|_{t=T} = D_t^5 u(x,t)|_{t=T} = 0, \ x \in \Omega. \tag{5}$$

The study of the solvability of boundary value problems for quasi-hyperbolic equations began, apparently, with the works of V.N. Vragov [1, 2]. Studies in [3–7] are related to further investigations of operators similar to $L$. One of the main conditions for correctness in these studies was the condition that the parameter $\lambda$ is non-negative. Investigations of nonlocal problems with integral conditions for linear parabolic equations, for differential equations of odd order, and for some classes of non-stationary equations have been actively carried out recently in the works of A.I. Kozhanov [4, 6, 7]. In [5], the solvability of problem (2), (3), (5) for fourth order quasi- hyperbolic equations with $p = 2$ is investigated. In the work [8] boundary value problems with normal derivatives were studied for elliptic equations of the $2l-$st order with constant real coefficients. For these problems, sufficient conditions for the Fredholm solvability of the problem are obtained and formulas for the index of this problem are given. An explicit form of the Green function of the Dirichlet problem for the model-polyharmonic equation $\Delta^l u = f$ in a multidimensional sphere was constructed in [9]. [10, 11] are devoted to an investigations of the solvability of various boundary value problems of order $0 \leq k_1 < k_2 < ... < k_l \leq 2l-1$ for the polyharmonic equation in a multidimensional ball.

In this paper, we describe calculation of eigenvalues $\lambda_m^{(1)}(\lambda_m^{(2)})$ of spectral problems $I_{3,\lambda}(II_{3,\lambda})$ for a sixth order quasi-hyperbolic equation and study solvability of boundary value problems $I_{3,\lambda}(II_{3,\lambda})$ for cases when $\lambda$ coincides or does not coincide with $\lambda_m^{(1)}(\lambda_m^{(2)})$.

*Supporting statement*

We denote by $V_3$ – the linear set of functions $v(x,t)$, belonging to the space $L_2(Q)$ and having generalized derivatives with respect to spatial variable up to the second order inclusively belonging to the same space and with respect to the variable $t$ up to the order 6 inclusively, with the norm

$$\|v\|_{V_3} = \left(\int_Q \Big[v^2 + \sum_{i,j=1}^{n}\Big(\frac{\partial^2 v}{\partial x_i \partial x_j}\Big)^2 + \Big(\frac{\partial^6 v}{\partial t^6}\Big)^2\Big]dxdt\right)^{\frac{1}{2}}.$$

Obviously, the space $V_3$ with this norm is a Banach space.

Let $v(x)$ be function from the space $\overset{\circ}{W}{}_2^1(\Omega)$. The following inequality is true

$$\int_\Omega v^2(x)dx \leq c_0 \int_\Omega \sum_{i=1}^{n} v_{x_i}^2(x)dx, \tag{6}$$

where constant $c_0$ defined only by area $\Omega$ (see, example [12]).

For the function from the space $V_3$ satisfying condition (3), the following inequality holds:

$$\int_\Omega v^2(x,t_0)dx \leq T^3 \int_0^T \int_\Omega v_{ttt}^2(x,t)dxdt, \quad t_0 \in [0,T], \tag{7}$$

$$\int_0^T \int_\Omega v^2(x,t)dxdt \leq \frac{T^6}{8} \int_0^T \int_\Omega v_{ttt}^2(x,t)dxdt. \tag{8}$$

Let $\omega_j(x)$ be the eigenfunction of the Dirichlet problem for the Laplace operator corresponding to the eigenvalue $\mu_j$:

$$\Delta\omega_j(x) = \mu_j\omega_j(x), \quad \omega_j(x)|_\Gamma = 0.$$

*3 Main results*

*Theorem 1.* Let $\lambda > c_1, c_1 = \min\{-\frac{1}{c_0}, -\frac{40}{T^6}\}$, $c_0$ from (6). Then the homogeneous boundary value problem $I_{3,\lambda}$ has only zero solution in the space $V_3$. On the interval $(-\infty, c_1)$ there exists a countable set of numbers $\lambda_m^{(1)}$ such that for $\lambda = \lambda_m^{(1)}$ the homogeneous boundary value problem $I_{3,\lambda}$ has a non-trivial solution.

*Proof.* First, we prove the uniqueness of the solution to the problem $I_{3,\lambda}$. Let $A > T$. We consider the equality

$$\int_0^T \int_\Omega (A-t)Lu \cdot u_t dxdt = 0.$$

Integrating by parts and using conditions (2), (3) we get

$$\frac{A-T}{2}\int_\Omega [u_{ttt}^2(x,T) + \sum_{i=1}^{n} u_{x_i}^2(x,T)]dx + \frac{5}{2}\int_0^T \int_\Omega u_{ttt}^2 dxdt +$$

$$+\frac{1}{2}\sum_{i=1}^{n}\int_0^T \int_\Omega u_{x_i}^2 dxdt = -\frac{\lambda(A-T)}{2}\int_\Omega u^2(x,T)dx - \frac{\lambda}{2}\int_0^T \int_\Omega u^2 dxdt = I. \tag{9}$$

When $\lambda \geq 0$ it follows from this equality that $u(x,t) \equiv 0$.

We now consider the case of negative values of $\lambda$. On the one hand due to expressions (6) and (7), there is an inequality

$$|I| = |-\frac{\lambda(A-T)}{2}\int_\Omega u^2(x,T)dx - \frac{\lambda}{2}\int_0^T\int_\Omega u^2 dxdt| \leq$$

$$\leq \frac{|\lambda|(A-T)}{2}T^3\int_0^T\int_\Omega u_{ttt}^2 dxdt + \frac{|\lambda|}{2}c_0\sum_{i=1}^n\int_0^T\int_\Omega u_{x_i}^2 dxdt. \tag{10}$$

On the other hand, due to inequalities (7) and (8) we get

$$|I| \leq \frac{|\lambda|(A-T)}{2}T^3\int_0^T\int_\Omega u_{ttt}^2 dxdt + \frac{|\lambda|T^6}{2\cdot 2^3}\int_0^T\int_\Omega u_{ttt}^2 dxdt.$$

If $c_1 = -\frac{1}{c_0}$, then by evaluating the right side of (9) by (10), we get

$$\frac{A-T}{2}\int_\Omega[u_{ttt}^2(x,T) + \sum_{i=1}^n u_{x_i}^2(x,T)]dx +$$

$$+\frac{5-|\lambda|(A-T)T^3}{2}\int_0^T\int_\Omega u_{ttt}^2 dxdt + \frac{1-|\lambda|c_0}{2}\sum_{i=1}^n\int_0^T\int_\Omega u_{x_i}^2 dxdt \leq 0. \tag{11}$$

Since inequality $|\lambda|c_0 < 1$ holds and we can choose number $A$ close to number $T$, the inequality

$$5 - |\lambda|(A-T)T^3 > 0$$

holds for fixed values of $\lambda$. Then, from (11) it follows that $u(x,t) \equiv 0$.

In the case of $c_1 = -\frac{40}{T^6}$, we have

$$\frac{A-T}{2}\int_\Omega[u_{ttt}^2(x,T) + \sum_{i=1}^n u_{x_i}^2(x,T)]dx +$$

$$+\frac{40-8|\lambda|(A-T)T^3-|\lambda|T^6}{2\cdot 2^3}\int_0^T\int_\Omega u_{ttt}^2 dxdt + \frac{1}{2}\sum_{i=1}^n\int_0^T\int_\Omega u_{x_i}^2 dxdt \leq 0. \tag{12}$$

Since $40 - |\lambda|T^6 > 0$, then choosing again $A$ close to the $T$,

$$40 - 8|\lambda|(A-T)T^3 - |\lambda|T^6 > 0$$

inequality can be achieved. Then, from (12) we also get $u(x,t) \equiv 0$.

The solution to equation (1) is sought in the form $u(x,t) = \varphi(t)\omega_j(x)$. Then function $\varphi(t)$ must be the solution to equation

$$-D_t^6\varphi(t) + [\mu_j - \lambda]\varphi(t) = 0, \tag{13}$$

satisfying condition

$$\varphi(0) = \varphi'(0) = \varphi''(0) = \varphi'''(0) = \varphi'(T) = \varphi''(T) = 0. \tag{14}$$

*a*) If $\mu_j - \lambda > 0$, then general solution (13) has the form

$$\varphi(t) = C_1e^{\gamma_j t} + C_2e^{\frac{\gamma_j t}{2}}cos\frac{\sqrt{3}}{2}\gamma_j t + C_3e^{\frac{\gamma_j t}{2}}sin\frac{\sqrt{3}}{2}\gamma_j t +$$

$$+C_4e^{-\gamma_j t} + C_5e^{-\frac{\gamma_j t}{2}}cos\frac{\sqrt{3}}{2}\gamma_j t + C_6e^{-\frac{\gamma_j t}{2}}sin\frac{\sqrt{3}}{2}\gamma_j t, \tag{15}$$

where $\gamma_j = (\mu_j - \lambda)^{\frac{1}{6}}$. Taking in account (14), the numbers $C_j, j = \overline{1,6}$, should be a solution to an algebraic system

$$\begin{cases} C_1+C_2+C_4+C_5=0, \\ C_1+\frac{1}{2}C_2+\frac{\sqrt{3}}{2}C_3-C_4-\frac{1}{2}C_5+\frac{\sqrt{3}}{2}C_6=0, \\ C_1-\frac{1}{2}C_2+\frac{\sqrt{3}}{2}C_3+C_4-\frac{1}{2}C_5-\frac{\sqrt{3}}{2}C_6=0, \\ C_1-C_2-C_4+C_5=0, \\ E^2C_1+E(\frac{1}{2}C-\frac{\sqrt{3}}{2}S)C_2+E(\frac{\sqrt{3}}{2}C+\frac{1}{2}S)C_3- \\ E^{-2}C_4-E^{-1}(\frac{1}{2}C+\frac{\sqrt{3}}{2}S)C_5+E^{-1}(\frac{\sqrt{3}}{2}C-\frac{1}{2}S)C_6=0, \\ E^2C_1-E(\frac{1}{2}C+\frac{\sqrt{3}}{2}S)C_2+E(\frac{\sqrt{3}}{2}C-\frac{1}{2}S)C_3+ \\ E^{-2}C_4+E^{-1}(-\frac{1}{2}C+\frac{\sqrt{3}}{2}S)C_5-E^{-1}(\frac{\sqrt{3}}{2}C+\frac{1}{2}S)C_6=0, \end{cases}$$

where

$$E=e^{\frac{\gamma_j T}{2}}, C=cos\frac{\sqrt{3}}{2}\gamma_j T, S=sin\frac{\sqrt{3}}{2}\gamma_j T.$$

The determinant of this system will be equal to

$$D(\gamma_j)=\frac{3}{2}\left[2E^3C-3E^2-6EC+10+4C^2-6E^{-1}C-3E^{-2}+2E^{-3}C\right],$$

and it can not be zero, therefore, in this case, problem (13), (14) have not non-trivial solutions.

*b*) If $\mu_j-\lambda<0$, then general solution (13) has a form

$$\varphi(t)=C_1e^{\frac{\sqrt{3}}{2}\gamma_j t}cos\frac{\gamma_j t}{2}+C_2e^{\frac{\sqrt{3}}{2}\gamma_j t}sin\frac{\gamma_j t}{2}+C_3e^{-\frac{\sqrt{3}}{2}\gamma_j t}cos\frac{\gamma_j t}{2}+$$
$$+C_4e^{-\frac{\sqrt{3}}{2}\gamma_j t}sin\frac{\gamma_j t}{2}+C_5cos\gamma_j t+C_6sin\gamma_j t, \quad (16)$$

where $\gamma_j=(\lambda-\mu_j)^{\frac{1}{6}}$. Considering (14), the number $C_j, j=\overline{1,6}$, should be a solution to an algebraic system

$$\begin{cases} C_1+C_3+C_5=0, \\ \frac{\sqrt{3}}{2}C_1+\frac{1}{2}C_2-\frac{\sqrt{3}}{2}C_3+\frac{1}{2}C_4+C_6=0, \\ \frac{1}{2}C_1+\frac{\sqrt{3}}{2}C_2+\frac{1}{2}C_3-\frac{\sqrt{3}}{2}C_4-C_5=0, \\ C_2+C_4-C_6=0, \\ E(\frac{\sqrt{3}}{2}C-\frac{1}{2}S)C_1+E(\frac{1}{2}C+\frac{\sqrt{3}}{2}S)C_2-E^{-1}(\frac{\sqrt{3}}{2}C+\frac{1}{2}S)C_3+ \\ +E^{-1}(\frac{1}{2}C-\frac{\sqrt{3}}{2}S)C_4-2CSC_5+(C^2-S^2)C_6=0, \\ E(\frac{1}{2}C-\frac{\sqrt{3}}{2}S)C_1+E(\frac{\sqrt{3}}{2}C+\frac{1}{2}S)C_2+E^{-1}(\frac{1}{2}C+\frac{\sqrt{3}}{2}S)C_3+ \\ +E^{-1}(-\frac{\sqrt{3}}{2}C+\frac{1}{2}S)C_4+(-C^2+S^2)C_5-2CSC_6=0, \end{cases}$$

where $E=e^{\frac{\sqrt{3}}{2}\gamma_j T}, C=cos\frac{\gamma_j T}{2}, S=sin\frac{\gamma_j T}{2}$.

This system has a nontrivial solution if the determinant

$$D(\gamma_j)=-C^2S^2=-\frac{1}{4}sin^2\gamma_j T=0, \quad (17)$$

is equal to zero. From (17) we get desired set of eigenvalues

$$\lambda_{jk}^{(1)}=\mu_{jk}+\left(\frac{k\pi}{T}\right)^6, \quad k=1,2,... \quad (14)$$

The theorem 1 is proved.

*Consequence 1.* The problem $I_{3,\lambda}$ does not have real eigenvalues other than the numbers $\lambda_{jk}^{(1)}$ from (18) and the family $\{\lambda_{jk}^{(1)}\}_{j,k=1}^{\infty}$ does not have finite limit points. All eigenvalues of $\{\lambda_{jk}^{(1)}\}_{j,k=1}^{\infty}$ are finite multiplicity.

*Proof.* The fact that the problem $I_{3,\lambda}$ does not have real eigenvalues other than the numbers $\lambda_{jk}^{(1)}$, follows from the basis of the system of functions

$$\{\omega_j(x)\}_{j=1}^{\infty}$$

in space $W_2^2(\Omega)$.

Suppose that the family $\{\lambda_{jk}^{(1)}\}_{j,k=1}^{\infty}$ has a finite limit point. Then there is a family $(j_i,k_i)$ of pairs of natural numbers such that $j_i+k_i\to\infty$ such $i\to\infty$ and the sequence $\lambda_{jk}^{(1)}$ will be fundamental. Note that the indices

$j_i$, cannot be limited together, since in this case $\lambda_{jk} = \mu_{jk} + \left(\frac{k\pi}{T}\right)^6$, $k = 1, 2, ...,$ which cannot be true for a fundamental sequence.

Further, the indices $k_i$ also cannot be limited together, since in this case the sequence $\{\mu_{j_i} - \mu_{j_{i+m}}\}$, will be limited, which is not the case. Therefore, for the indices $j_i$ and $k_i$, $j_i \to \infty$, $k_i \to \infty$ hold for $i \to \infty$. But then $\lambda_{j_k k_i} \to -\infty$, which again does not hold for a fundamental sequence. From the above, the validity of second part of consequence follows. The finite multiplicity of each eigenvalue $\lambda_{jk}^{(1)}$ follows from the fact that for fixed numbers j and k the equality $\lambda_{jk}^{(1)} = \lambda_{j_1 k_1}^{(1)}$ is only possible for a finite set of indices $j_1$ and $k_1$. Consequence is proved.

Note that for the case $n = 1$ the eigenvalues $\mu_j$ could be in exact form, and then it is easy to give constructive conditions for the simplicity of each eigenvalue $\lambda_{jk}^{(1)}$ or to provide examples in which the eigenvalues will have a multiplicity greater than one. In the general case, it is also easy to give simplicity conditions, but it seems that they will not be constructive.

*Consequence 2.* The eigenvalues $\lambda_{jk}^{(1)}$ of the problem $I_{3,\lambda}$ correspond to the eigenfunctions

$$u_{jk}^{(1)}(x,t) = \omega_j(x)\varphi_k^{(1)}(t),$$

where function $\varphi_k^{(1)}(t)$ represented as

$$\varphi_k^{(1)}(t) = \frac{C}{12S_k(E_k - E_k^{-1})}\Big[-(3C_k(E_k - E_k^{-1}) + 5\sqrt{3}S_k(E_k + E_k^{-1}) + 6)e^{\frac{\sqrt{3}}{2}\gamma_k t}cos\frac{\gamma_k t}{2} -$$
$$(3\sqrt{3}C_k(E_k + E_k^{-1}) - 15S_k(E_k - E_k^{-1}) + 3\sqrt{3})e^{\frac{\sqrt{3}}{2}\gamma_k t}sin\frac{\gamma_k t}{2} +$$
$$(-3C_k(E_k + E_k^{-1}) + (4 + 5\sqrt{3})S_k(E_k - E_k^{-1}) - 6)e^{-\frac{\sqrt{3}}{2}\gamma_k t}cos\frac{\gamma_k t}{2} +$$
$$(3\sqrt{3}C_k(E_k + E_k^{-1}) + 15S_k(E_k - E_k^{-1}) - 6\sqrt{3})e^{-\frac{\sqrt{3}}{2}\gamma_k t}sin\frac{\gamma_k t}{2} +$$
$$(6C_k(E_k + E_k^{-1}) - 6\sqrt{3}S_k(E_k - E_k^{-1}) + 12)cos\gamma_k t + 12S_k(E_k - E_k^{-1})sin\gamma_k t\Big],$$

$$E_k = e^{\frac{\sqrt{3}\pi k}{2}},\ C_k = cos\frac{\pi k}{2},\ S_k = sin\frac{\pi k}{2},\ C = Const,\quad k = 1, 2, ...$$

Now consider the problem $II_3$. The study of problem $II_3$ is similar to $I_3$. The following theorem holds.

*Theorem 2.* For $\lambda > c_1, c_1 = \min\{-\frac{1}{c_0}, -\frac{40}{T^6}\}$, the homogeneous boundary problem $II_{3,\lambda}$ has only zero solution in the space $V_3$. On the interval $(-\infty, c_1)$ there doesn't exist a countable set of the numbers $\lambda_m^{(2)}$ such that for $\lambda = \lambda_m^{(2)}$ homogeneous boundary problem $II_{3,\lambda}$ has only trivial solution.

The solution to equation (1) is sought in the form $u(x,t) = \varphi(t)\omega_j(x)$. Then, function $\varphi(t)$ must be solution to equation (13) that satisfy conditions

$$\varphi(0) = \varphi'(0) = \varphi''(0) = \varphi'''(0) = \varphi''''(T) = \varphi'''''(T) = 0. \tag{19}$$

*a*) If $\mu_j - \lambda > 0$, then general solution $\varphi(t)$ has a form

$$\varphi(t) = C_1e^{\gamma_j t} + C_2e^{\frac{\gamma_j t}{2}}cos\frac{\sqrt{3}}{2}\gamma_j t + C_3e^{\frac{\gamma_j t}{2}}sin\frac{\sqrt{3}}{2}\gamma_j t +$$
$$+C_4e^{-\gamma_j t} + C_5e^{-\frac{\gamma_j t}{2}}cos\frac{\sqrt{3}}{2}\gamma_j t + C_6e^{-\frac{\gamma_j t}{2}}sin\frac{\sqrt{3}}{2}\gamma_j t,$$

where $\gamma_j = (\mu_j - \lambda)^{\frac{1}{6}}$. Considering (15), $C_j, j = \overline{1,6}$, should be a solution to an algebraic system

$$\begin{cases} C_1 + C_2 + C_4 + C_5 = 0, \\ C_1 + \frac{1}{2}C_2 + \frac{\sqrt{3}}{2}C_3 - C_4 - \frac{1}{2}C_5 + \frac{\sqrt{3}}{2}C_6 = 0, \\ C_1 - \frac{1}{2}C_2 + \frac{\sqrt{3}}{2}C_3 + C_4 - \frac{1}{2}C_5 - \frac{\sqrt{3}}{2}C_6 = 0, \\ C_1 - C_2 - C_4 + C_5 = 0, \\ E^2C_1 + E(-\frac{1}{2}C + \frac{\sqrt{3}}{2}S)C_2 - E(\frac{\sqrt{3}}{2}C + \frac{1}{2}S)C_3 + \\ E^{-2}C_4 - E^{-1}(\frac{1}{2}C + \frac{\sqrt{3}}{2}S)C_5 + E^{-1}(\frac{\sqrt{3}}{2}C - \frac{1}{2}S)C_6 = 0, \\ E^2C_1 + E(\frac{1}{2}C + \frac{\sqrt{3}}{2}S)C_2 + E(-\frac{\sqrt{3}}{2}C + \frac{1}{2}S)C_3 - \\ E^{-2}C_4 + E^{-1}(-\frac{1}{2}C + \frac{\sqrt{3}}{2}S)C_5 - E^{-1}(\frac{\sqrt{3}}{2}C + \frac{1}{2}S)C_6 = 0, \end{cases}$$

where $E = e^{\frac{\gamma_j T}{2}}, C = cos\frac{\sqrt{3}}{2}\gamma_j T, S = sin\frac{\sqrt{3}}{2}\gamma_j T$. The determinant of this system will be equal to

$$D(\gamma_j) = -\frac{3}{2}\left[2E^3C + 3E^2 + 6EC + 10 + 4C^2 + 6E^{-1} + 3E^{-2} + 2E^{-3}C\right],$$

and it can not be zero, therefore, in this case, there are no non-trivial solutions.

*b*) If $\mu_j - \lambda < 0$, then function $\varphi(t)$ has a form

$$\varphi(t) = C_1 e^{\frac{\sqrt{3}}{2}\gamma_j t} cos\frac{\gamma_j t}{2} + C_2 e^{\frac{\sqrt{3}}{2}\gamma_j t} sin\frac{\gamma_j t}{2} + C_3 e^{-\frac{\sqrt{3}}{2}\gamma_j t} cos\frac{\gamma_j t}{2} +$$

$$C_4 e^{-\frac{\sqrt{3}}{2}\gamma_j t} sin\frac{\gamma_j t}{2} + C_5 cos\gamma_j t + C_6 sin\gamma_j t,$$

where $\gamma_j = (\lambda - \mu_j)^{\frac{1}{6}}$. In this case, $C_j, j = \overline{1,6}$, should be a solution to an algebraic system

$$\begin{cases} C_1 + C_3 + C_5 = 0, \\ \frac{\sqrt{3}}{2}C_1 + \frac{1}{2}C_2 - \frac{\sqrt{3}}{2}C_3 + \frac{1}{2}C_4 + C_6 = 0, \\ \frac{1}{2}C_1 + \frac{\sqrt{3}}{2}C_2 + \frac{1}{2}C_3 - \frac{\sqrt{3}}{2}C_4 - C_5 = 0, \\ C_2 + C_4 - C_6 = 0, \\ -E(\frac{1}{2}C + \frac{\sqrt{3}}{2}S)C_1 + E(\frac{\sqrt{3}}{2}C - \frac{1}{2}S)C_2 + E^{-1}(-\frac{1}{2}C + \frac{\sqrt{3}}{2}S)C_3 - \\ E^{-1}(\frac{\sqrt{3}}{2}C + \frac{1}{2}S)C_4 + (C^2 - S^2)C_5 + 2CSC_6 = 0, \\ -E(\frac{\sqrt{3}}{2}C + \frac{1}{2}S)C_1 + E(\frac{1}{2}C - \frac{\sqrt{3}}{2}S)C_2 + E^{-1}(\frac{\sqrt{3}}{2}C - \frac{1}{2}S)C_3 + \\ +E^{-1}(\frac{1}{2}C + \frac{\sqrt{3}}{2}S)C_4 - 2CSC_5 + (C^2 - S^2)C_6 = 0, \end{cases}$$

where $E = e^{\frac{\sqrt{3}}{2}\gamma_j T}, C = cos\frac{\gamma_j T}{2}, S = sin\frac{\gamma_j T}{2}$. The determinant of this system will be equal to

$$D(\gamma_j) = \frac{3}{4}\left[E^2 + 8EC^3 + 6 + 12C^2 + 8E^{-1}C^3 + E^{-2}\right],$$

also can not be zero.

In conclusion, the problem $II_{3,\lambda}$ does not have real eigenvalues $\lambda_{jk}^{(2)}$. The theorem 2 is proved.

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## Алтыншы ретті классикалық емес дифференциалдық теңдеуге арналған спектрлік есеп

Мақалада С.Л. Соболев кеңістігінде алтыншы ретті квазигиперболалық теңдеу үшін шеттік есептердің

$$Lu = -D_t^6 u + \Delta u - \lambda u$$

($D_t = \frac{\partial}{\partial t}$, $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ – Лаплас операторы, $\lambda$ – нақты параметр) тиянақты шешімділігі зерттелген. Берілген $L$ операторы үшін классикалық емес дифференциалдық теңдеуге екі спектрлік есеп қарастырылған. Қойылған есептің шешімінің жалғыздығы, бірінші есептің меншікті мәндері мен функцияларының бар екендігі дәлелденген, яғни бұл есептің нөлдік емес шешімдері табылған. Авторлар $\lambda < 0$ үшін $Lu = 0$ және теңдеудің біртектілік шарты орындалғанда спектрлік есептің меншікті функцияларының нөлден өзгеше шешімдер жүйесінің бар екендігін көрсетеді. Әдетте, $L$ операторы қарапайым гиперболалық оператор болғанда, мұндай қасиет орындалмайды.

*Кілт сөздер:* алтыншы ретті квазигиперболалық теңдеу, меншікті мәндер, меншікті функциялар, нөлдік емес шешімдер.

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## Спектральная задача для неклассических дифференциальных уравнений шестого порядка

В статье исследована корректная разрешимость краевых задач для квазигиперболического уравнения шестого порядка в пространстве С.Л. Соболева

$$Lu = -D_t^6 u + \Delta u - \lambda u,$$

$D_t = \frac{\partial}{\partial t}$, $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ – оператор Лапласа; $\lambda$ – вещественный параметр. Рассмотрены две неклассические спектральные задачи для данного оператора $L$ и устанавливается единственность поставленных задач. Вычислены собственные значения и собственные функции поставленной первой задачи для квазигиперболического уравнения шестого порядка. Авторами показано, что уравнение $Lu = 0$ при $\lambda < 0$ и при выполнении однородных условий обладает счетным множеством нетривиальных решений. Обычно такой факт не имеет места, когда оператор $L$ есть обычный гиперболический оператор.

*Ключевые слова:* квазигиперболические уравнения шестого порядка, собственные значения, собственные функции, нетривиальные решения.

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# Green function method for a fractional–order delay differential equation

In this paper, we investigated a boundary value problem with the Sturm-Liouville type conditions for a linear ordinary differential equation of fractional order with delay. The condition for the unique solvability of the problem is obtained in the form $\triangle \neq 0$. The Green function of the problem, in terms of which the solution of the boundary value problem under study is written out, is constructed. The existence and uniqueness theorem for the solution of the problem is proved. It is also showed that in the case when the condition of unique solvability is violated, i.e $\triangle = 0$, then the solution of the boundary value problem is not unique. Using the notation of the generalized Mittag-Leffler function via the generalized Wright function, we also studied the properties of the function $\triangle$ as $\lambda \to \infty$ and $\lambda \to -\infty$. Using asymptotic formulas for the generalized Wright function, a theorem on the finiteness of the number of eigenvalues of a boundary value problem with the Sturm-Liouville type conditions is proved.

*Keywords:* Fractional differential equation, delay differential equation, Green function, generalized Mittag-Leffler function, generalized Wright function.

*Introduction*

Consider the equation

$$\frac{d^\alpha}{dt^\alpha}u(t) - \lambda u(t) - \mu H(t-\tau)u(t-\tau) = f(t), \quad 0 < t < 1, \tag{1}$$

where $\frac{d^\alpha}{dt^\alpha}$ is the Riemann-Liouville fractional derivative [1], $1 < \alpha \leq 2$, $\lambda, \mu$ are the arbitrary constants, $\tau$ is the fixed positive number, $H(t)$ denotes the Heaviside function.

At present, the number of studies on fractional calculation has noticeably increased. This is due to the fact that fractional order differential equations are used in mathematical modeling of processes that occur in various fields of natural science, such as physics, chemistry, biology, sociology, etc.

The most general references to the theory of fractional calculus one can find in [2–5] (see also the references in these works). A linear ordinary differential equation of fractional order was considered by Barrett [6] in 1954. Existence and uniqueness theorem for a fractional-order differential equation is proved in [7] by Dzhrbashyan and Nersesyan. Sturm-Liuville type boundary value problem for fractional differential operator was investigated by Dzhrbashyan in [8]. The initial value problem for a linear ordinary differential equation of fractional order was studied by Pskhu in [9].

Significant works were devoted to the delay differential equations (difference-differential equations) by Norkin in [10], Bellman and Cooke in [11], Elsgolts and Norkin in [12], Myshkis [13], Hale in 1977 [14].

The initial-value problem and the problem with general linear two-point boundary conditions, the Dirichlet and the Neumann problems for linear ordinary differential equation with Caputo derivative with delay in [15–17] respectevely were solved.

The Cauchy problem for Eq.(1) was considered in [18], and the solutions to the Dirichlet and the Neumann problems were obtained in [19].

In this paper, we construct the Green function of the Sturm-Liouville type boundary value problem for Eq.(1) and prove the finiteness theorem for the number of real eigenvalues of the study problem.

*Auxiliary*

The Riemann-Liouville fractional operator is define by the formula

$$\frac{d^\alpha}{dt^\alpha}u(t) = D^\alpha_{at}u(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int\limits_a^t \frac{u(\xi)d\xi}{(t-\xi)^{\alpha-n+1}}, \quad n-1<\alpha\leq n, n\in\mathbb{N},$$

where $\Gamma(z) = \int\limits_0^\infty e^{-t}t^{z-1}dt$ is the Euler gamma function.

The Mittag-Leffler function is given by the power series [20]

$$E_{\alpha,\beta}(z) = \sum_{k=o}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},$$

and the generalized Mittag-Leffler function defines by the series [21]

$$E^\rho_{\alpha,\beta} = \sum_{k=o}^{\infty}\frac{(\rho)_k z^k}{\Gamma(\alpha k+\beta)k!},$$

where $(\rho)_k = \frac{\Gamma(\rho+k)}{\Gamma(\rho)}$ is the Pokhhammer symbol. The generalized Mittag-Leffler function reduces to $E_{\alpha,\beta}(z)$ when we set $\rho = 1$.

Consider the function

$$W_\nu(t)=W_\nu(t,\tau;\lambda,\mu)=\sum_{m=0}^{\infty}\mu^m(t-m\tau)_+^{\alpha m+\nu-1}E^{m+1}_{\alpha,\alpha m+\nu}(\lambda(t-m\tau)^\alpha_+), \nu\in\mathbb{R}, \quad (2)$$

where

$$(t-m\tau)_+ = \begin{cases} t-m\tau, & t-m\tau>0,\\ 0, & t-m\tau\leq 0.\end{cases}$$

It follows from (2) that

$$W_k^{(i)}(0) = \begin{cases} 0, k\neq i+1,\\ 1, k=i+1.\end{cases} \quad (3)$$

*Remark 1.* For some $m$ the expression $t-m\tau<0$, therefore the series in (2) contains a finite number of terms $N\leq[\frac{t}{\tau}]+1$.

Function (2) satisfies the following properties [16]

$$D^\alpha_{0t}W_\nu(t) = W_{\nu-\alpha}(t), \quad \alpha\in\mathbb{R}, \quad \nu>0, \quad (4)$$

$$W_\nu(t) = \lambda W_{\nu+\alpha}(t) + \mu W_{\nu+\alpha}(t-\tau) + \frac{t^{\nu-1}}{\Gamma(\nu)}, \qquad \alpha>0, \quad \nu\in\mathbb{R}, \quad (5)$$

which are clear by the formula of differentiation [21]

$$\frac{d^m}{dz^m}\left(z^{\beta-1}E^\rho_{\alpha,\beta}(z^\alpha)\right) = z^{\beta-m-1}E^\rho_{\alpha,\beta-m}(z^\alpha))$$

and by the autotransformation formula [22]:

$$E^\rho_{\alpha,\beta}(t) - E^{\rho-1}_{\alpha,\beta}(t) = tE^\rho_{\alpha,\alpha+\beta}(t)$$

of the generalized Mittag-Leffler function.

*Main results*

A function $u(t)$ is called *a regular solution* of Eq.(1) if $D_{0t}^{\alpha-2}u(t) \in C^2(0,1)$, $u(t) \in L(0,1)$ and $u(t)$ satisfies Eq. (1) for all $0 < t < 1$.

*The problem we solve here is to find the regular solution to equation* (1) *satisfying the conditions*

$$a\lim_{t\to 0} D_{0t}^{\alpha-1}u(t) + b\lim_{t\to 0} D_{0t}^{\alpha-2}u(t) = 0,$$
$$c\lim_{t\to 1} D_{0t}^{\alpha-1}u(t) + d\lim_{t\to 1} D_{0t}^{\alpha-2}u(t) = 0, \tag{6}$$

*where* $a^2 + b^2 \neq 0$ *and* $c^2 + d^2 \neq 0$.

*Green function*

Assume $G(t,\xi)$ is given by

$$G(t,\xi) = H(t-\xi)W_\alpha(t-\xi) + (cW_1(1-\xi) + dW_2(1-\xi))\frac{bW_\alpha(t) - aW_{\alpha-1}(t)}{\triangle} \tag{7}$$

with $\lambda$ and $\mu$ satisfying the following condition

$$\triangle = ac(\lambda W_\alpha(1) + \mu W_\alpha(1-\tau)) + (ad-bc)W_1(1) - bdW_2(1) \neq 0. \tag{8}$$

Here the function $W_\nu(t)$ is defined via (2).

We demonstrate the validity of the following properties for the function $G(t,\xi)$ (7).

*1. The function* $G(t,\xi)$ *is continuous for all values of* $t$ *and* $\xi$ *from the closed interval* $[0,1]$.

This property implies from relation (7) and condition (8).

*2. The function* $G(t,\xi)$ *satisfies the conditions*

$$\lim_{\varepsilon\to 0}[D_{0t}^{\alpha-2}G_\xi(t,\xi)|_{\xi=t+\varepsilon} - D_{0t}^{\alpha-2}G_\xi(t,\xi)|_{\xi=t-\varepsilon}] = 1. \tag{9}$$

Indeed,

$$D_{0t}^{\alpha-2}G_\xi(t,\xi) = -H(t-\xi)W_1(t-\xi)$$
$$-\Big(c\lambda W_\alpha(1-\xi) + c\mu W_\alpha(1-\xi-\tau) + dW_1(1-\xi)\Big)\frac{bW_2(t) - aW_1(t)}{\triangle}. \tag{10}$$

Insert (10) into (9) as $\xi = t+\varepsilon$ and $\xi = t-\varepsilon$. Passing to the limit as $\varepsilon \to 0$ we get the property (9).

*3. The function* $G(t,\xi)$ *is the solution to the equation*

$$\partial_{1\xi}^{\alpha}G(t,\xi) - \lambda G(t,\xi) - \mu H(1-\tau-\xi)G(t,\xi+\tau) = 0. \tag{11}$$

Here $\partial_{0t}^{\alpha}$ is the Caputo derivative [23; 11] defines as

$$\partial_{0t}^{\alpha}u(t) = D_{0t}^{\alpha-2}u''(t) = \frac{1}{\Gamma(2-\alpha)}\int_0^t \frac{u''(\xi)d\xi}{(t-\xi)^{\alpha-1}}.$$

This property implies the presentation of the function (7) and the relations (4), (5).

*4. The function* $G(t,\xi)$ *satisfies the boundary conditions*

$$\begin{cases} a\lim_{\xi\to 0} D_{0t}^{\alpha-2}G_\xi(t,\xi) + b\lim_{\xi\to 0} D_{0t}^{\alpha-2}G(t,\xi) = 0 \\ c\lim_{\xi\to 1} D_{0t}^{\alpha-2}G_\xi(t,\xi) + d\lim_{\xi\to 1} D_{0t}^{\alpha-2}G(t,\xi) = 0 \end{cases}. \tag{12}$$

This property obviously implies the relations (4), (5).

The function $G(t,\xi)$ that possesses properties 1-4 is called Green function for problem (1), (6).

*Existence and uniqueness theorem*

*Theorem 1. Assume the function* $f(t) \in L(0,1) \cap C(0,1)$ *and the condition* (8) *is satisfied. Then*
*1) there exists a regular solution to problem* (1), (6) *in the form of*

$$u(t) = \int_0^1 f(\xi)G(t,\xi)d\xi; \tag{13}$$

*2) the solution to problem* (1), (6) *is unique if and only if condition* (8) *is satisfied.*

*Proof.* First we illustrate that the solution to problem (1), (6) has the form (13). To clear this, multiply both sides of Eq. (1) (given in terms of variable $\xi$) by $D_{0t}^{\alpha-2}G(t,\xi)$ and integrate it with respect to variable $\xi$ ranging from $\varepsilon$ to $1-\varepsilon$ ($\varepsilon \to 0$):

$$\int_\varepsilon^{1-\varepsilon} D_{0t}^{\alpha-2}G(t,\xi)D_{0\xi}^{\alpha}u(\xi)d\xi - \lambda \int_\varepsilon^{1-\varepsilon} u(\xi)D_{0t}^{\alpha-2}G(t,\xi)d\xi$$

$$-\mu \int_\varepsilon^{1-\varepsilon} H(t-\tau)u(\xi-\tau)D_{0t}^{\alpha-2}G(t,\xi)d\xi = \int_\varepsilon^{1-\varepsilon} f(\xi)D_{0t}^{\alpha-2}G(t,\xi)d\xi, \quad 0<t<1. \tag{14}$$

Integrate by parts the first term of equality (14):

$$\int_\varepsilon^{1-\varepsilon} D_{0t}^{\alpha-2}G(t,\xi)D_{0\xi}^{\alpha}u(\xi)d\xi = D_{0t}^{\alpha-2}G(t,\xi)D_{0\xi}^{\alpha-1}u(\xi)\Big|_\varepsilon^{1-\varepsilon} - \int_\varepsilon^{t} D_{0t}^{\alpha-2}G_\xi(t,\xi)D_{0\xi}^{\alpha-1}u(\xi)d\xi$$

$$-\int_t^{1-\varepsilon} D_{0t}^{\alpha-2}G_\xi(t,\xi)D_{0\xi}^{\alpha-1}u(\xi)d\xi = D_{0t}^{\alpha-2}G(t,\xi)D_{0\xi}^{\alpha-1}u(\xi)\Big|_{\xi=1} - D_{0t}^{\alpha-2}G(t\xi)D_{0\xi}^{\alpha-1}u(\xi)\Big|_{\xi=0}$$

$$+D_{0t}^{\alpha-2}u(\xi)\left[D_{0t}^{\alpha-2}G_\xi(t,\xi)\Big|_{\xi=t+0} - D_{0t}^{\alpha-2}G_\xi(t,\xi)\Big|_{\xi=t-0}\right] + D_{0t}^{\alpha-2}G_\xi(t,\xi)D_{0\xi}^{\alpha-2}u(\xi)\Big|_{\xi=0}$$

$$-D_{0t}^{\alpha-2}G_\xi(t,\xi)D_{0\xi}^{\alpha-2}u(\xi)\Big|_{\xi=1} + \int_0^1 D_{0t}^{\alpha-2}G_{\xi\xi}(t,\xi)D_{0\xi}^{\alpha-2}u(\xi)d\xi. \tag{15}$$

Applying to (15) the properties (9), (12) of function (7) and conditions (6) of the problem we get the following formula

$$D_{0t}^{\alpha-2}u(\xi) + \int_0^1 D_{0\xi}^{\alpha-2}u(\xi)D_{0t}^{\alpha-2}G_{\xi\xi}(t,\xi)d\xi. \tag{16}$$

Replace $\xi$ with $\xi-\tau$ in the third integral on the left-hand side of the expression (14) to reduce it to

$$\int_0^1 H(\xi-\tau)u(\xi-\tau)G(t,\xi)d\xi = \int_0^1 H(1-\tau-\xi)u(\xi)G(t,\xi+\tau)d\xi. \tag{17}$$

Put (16) and (17) into Eq. (14) and using the formula for fractional integration by parts [20, p. 15]

$$\int_a^b g(s)D_{as}^{\alpha}h(s)ds = \int_a^b h(s)D_{bs}^{\alpha}g(s)ds,$$

arrive at identity

$$D_{0t}^{\alpha-2}u(\xi) + D_{0t}^{\alpha-2}\int_0^1 u(\xi)\left[D_{1\xi}^{\alpha-2}G_{\xi\xi}(t,\xi) - \lambda G(t,\xi) - \mu H(1-t-\xi)G(t,\xi+\tau)\right]d\xi =$$

$$= D_{0t}^{\alpha-2} \int\limits_0^1 f(\xi) G(t,\xi) d\xi.$$

Taking advantage of the third property of Green function $G(t,\xi)$ (11) and finding the derivative of order $D_{0t}^{2-\alpha}$ we arrive at representation (13).

Next, we show that the function (13) is the solution to problem (1), (6). Formula (13) can be written out in terms of function $W_\nu(t)$ in the form of bellow:

$$u(t) = \int\limits_0^t f(\xi) W_\alpha(t-\xi) d\xi + \frac{bW_\alpha(t) - aW_{\alpha-1}(t)}{\triangle} \int\limits_0^1 f(\xi) \left(cW_1(1-\xi) + dW_2(1-\xi)\right) d\xi.$$

Next, using formulas (4), (5) obtain by the previous relation that

$$D_{0t}^{\alpha} u(t) = f(t) + \lambda \int\limits_0^1 f(\xi) G(t,\xi) d\xi + \mu \int\limits_0^1 f(\xi) G(t, \xi - \tau) d\xi.$$

Prove that the solution $u(t)$ satisfies the boundary conditions (6) (in view of relation (3)):

$$a \lim_{t\to 0} D_{0t}^{\alpha-1} u(t) + b \lim_{t\to 0} D_{0t}^{\alpha-2} u(t) = \frac{1}{\triangle} \int\limits_0^1 f(\xi) \left[cW_1(1-\xi) + dW_2(1-\xi)\right]$$

$$\times \left[abW_1(0) - a^2 \lambda W_\alpha(0) - a^2 \mu W_\alpha(-\tau) + b^2 W_2(0) - abW_1(0)\right] d\xi = 0;$$

$$c \lim_{t\to 1} D_{0t}^{\alpha-1} u(t) + d \lim_{t\to 1} D_{0t}^{\alpha-2} u(t) = \int\limits_0^1 f(\xi) [cW_1(1-\xi) + dW_2(1-\xi)] \times$$

$$\times \left[1 + \frac{-ac(\lambda W_\alpha(1) + \mu W_\alpha(1-\tau)) - (ad - cb) W_1(1) + bdW_2(1)}{\triangle}\right] d\xi =$$

$$= \int\limits_0^1 f(\xi) (cW_1(1-\xi) + dW_2(1-\xi)) \left(1 - \frac{\triangle}{\triangle}\right) d\xi = 0.$$

The task is now to show that if the condition (8) is not satisfied

$$\triangle = 0,$$

then the solution of the problem is not unique. Consider the function

$$\bar{u}(t) = C_1 W_\alpha(t) + C_2 W_{\alpha-1}(t),$$

which is the solution to the problem

$$\begin{gathered} D_{0t}^{\alpha} \bar{u}(t) - \lambda \bar{u}(t) - \mu H(t-\tau) \bar{u}(t-\tau) = 0, \\ a \lim_{t\to 0} D_{0t}^{\alpha-1} \bar{u}(t) + b \lim_{t\to 0} D_{0t}^{\alpha-2} \bar{u}(t) = 0, \\ c \lim_{t\to 1} D_{0t}^{\alpha-1} \bar{u}(t) + d \lim_{t\to 1} D_{0t}^{\alpha-2} \bar{u}(t) = 0. \end{gathered} \tag{18}$$

The conditions (18) can be written out in the form

$$\begin{gathered} aC_1 + bC_2 = 0, \\ C_1[W_1(1) + dW_2(1)] + C_2[c\lambda W_\alpha(1) + c\mu W_\alpha(1-\tau) + dW_1(1)] = 0. \end{gathered} \tag{19}$$

Then the determinant of the system (19) is equal to

$$\overline{\triangle} = \begin{vmatrix} a & b \\ cW_1(1) + dW_2(1) & c\lambda W_\alpha(1) + c\mu W_\alpha(1-\tau) + dW_1(1) \end{vmatrix} = 0.$$

Thus, solution to problem (1), (6) is unique if and only if condition (8) is satisfied.

*Remark.* For all

$$\lambda, \mu > 0, \quad (a-b)(c+d) > 0$$

condition (8) is always satisfied.

*On the finiteness of the number of real eigenvalues*

*Definition.* The eigenvalues of problem (1), (6) are the values $\lambda$, such that problem (1), (6) has a regular solution that is not the identically zero.

The set of real eigenvalues for problem (1), (6) coincides with the set of real zeros for the function

$$\Phi(\lambda) = ac(\lambda W_\alpha(1) + \mu W_\alpha(1-\tau)) + (ad-bc)W_1(1) - bdW_2(1). \tag{20}$$

*Theorem 2. Problem* (1), (6) *has only a finite number of real eigenvalues.*

The function $W_\nu(\lambda)$ can be written out as [2, p. 45]

$$W_\nu(1,\tau;\lambda,\mu) = \sum_{m=0}^{\infty} \frac{\mu^m}{m!}(1-m\tau)_+^{\alpha m+\nu-1}\, {}_1\Psi_1\left[\begin{matrix}(m+1,1)\\(\alpha m+\nu,\alpha)\end{matrix}\middle| \lambda(1-m\tau)_+^\alpha\right],$$

where

$${}_p\Psi_q\left[\begin{matrix}(a_l,\alpha_l)_{1,p}\\(b_l,\beta_l)_{1,q}\end{matrix}\middle| z\right] = \sum_{k=0}^{\infty} \frac{\prod_{l=1}^{p}\Gamma(a_l+\alpha_l k)}{\prod_{l=1}^{q}\Gamma(b_l+\beta_l k)}\frac{z^k}{k!}$$

is the generalized Wright function [24].

Function (20) is an integer function of parameter $\lambda$. Let us investigate the properties of the function (20) as $\lambda \to +\infty$ and $\lambda \to -\infty$.

As $\lambda \to +\infty$ the following asymptotic formula holds true for the generalized Wright function [24], [25]:

$${}_1\Psi_1\left[\begin{matrix}(m+1,1)\\(\alpha m+\nu,\alpha)\end{matrix}\middle|\lambda(1-m\tau)_+^\alpha\right] = \alpha^{-m}\lambda^{\frac{m(1-\alpha)-\nu+1}{\alpha}}(1-m\tau)_+^{m(1-\alpha)-\nu+1}e^{\lambda^{1/\alpha}(1-m\tau)_+}\left[1+O\left(\frac{1}{\lambda^{\frac{1}{\alpha}}}\right)\right].$$

Let $N$ be the maximum value of $m$ that satisfies the inequality $(1-m\tau)>0$. Then the asymptotic formula for function (20) is in the form

$$\Phi(\lambda) = \sum_{m=0}^{\infty}\frac{\mu^m\alpha^{-m}}{m}\lambda^{\frac{m-\alpha m+1}{\alpha}}\left[(1-m\tau)_+^m e^{\lambda^{1/\alpha}(1-m\tau)_+}\left(ac+(ad-bc)\lambda^{-1/\alpha}-bd\lambda^{-2/\alpha}\right)\right.$$
$$\left. + ac\frac{\mu}{\lambda}(1-(m+1)\tau)_+^m e^{\lambda^{1/\alpha}(1-(m+1)\tau)}\right] \times \left[1+O(\lambda^{-1/\alpha})\right].$$

Hence, as $\lambda \to \infty$, the series above increases without limit.

The asymptotic formula for the generalized Wright function as $\lambda \to -\infty$ has form [24], [25]

$${}_1\Psi_1\left[\begin{matrix}(m+1,1)\\(\alpha m+\nu,\alpha)\end{matrix}\middle|\lambda(1-m\tau)_+^\alpha\right] = \sum_{l=0}^{n}\frac{(-1)^{m+l+1}(l+m)!(1-m\tau)_+^{-\alpha(m+l+1)}}{|\lambda|^{m+l+1}\Gamma(\nu-\alpha-\alpha l)(m+l+1)!}$$
$$+O\left(\frac{1}{|\lambda|^m}\right).$$

Therefore

$$\Phi(\lambda) = \sum_{m=0}^{N}\frac{(-1)^{m+1}\mu^m}{(m+1)!|\lambda|^m}\left[\frac{ac}{\Gamma(-\alpha)}\left((1-m\tau)_+^{-1}+\frac{\mu(1-(m+1)_+\tau)^{-1}}{|\lambda|}\right)\right.$$
$$\left.+(ad-bc)\frac{(1-m\tau)^{-\alpha}}{|\lambda|\Gamma(1-\alpha)} - bd\frac{(1-m\tau)^{1-\alpha}}{|\lambda|\Gamma(2-\alpha)} + O\left(\frac{1}{|\lambda|^{N+1}}\right)\right]. \tag{21}$$

Consider the limit relation in the case when $\mu \neq 0$

$$\lim_{\lambda\to-\infty}\lambda^N\Phi(\lambda) = \frac{ac(-1)^{N+1}\mu^N(1-N\tau)_+^{-1}}{\Gamma(-\alpha)(N+1)!} \neq 0. \tag{22}$$

As $\mu = 0$ we have

$$\lim_{\lambda\to-\infty} \lambda\Phi(\lambda) = -\frac{ac}{\Gamma(-\alpha)}. \tag{23}$$

Since $\Phi(\lambda)$ is an entire function of the variable $\lambda$, it follows from relations (21), (22), and (23) that the series (20) may have only a finite number of real zeros. This establishes the theorem.

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М.Г. Мажгихова

## Бөлшек ретті кешікпелі аргументті дифференциалдық теңдеу үшін Грин функциясы әдісі

Мақалада кәдімгі сызықтық тұрақты коэффициентті кешікпелі аргументті бөлшек ретті дифференциалдық теңдеу үшін Штурм-Лиувилль типті шеттік есеп зерттелген. Қойылған есептің бірмәнді шешілуі $\triangle \neq 0$ түрінде алынды. Зерттеліп отырған есепті шешу үшін Грин әдісі қолданылды. Грин функциялары Миттаг-Леффлер жалпыланған функциялары терминінде жазылды. Зерттеліп отырған есептің шешуінің бар болуы және жалғыздығы жайлы теорема дәлелденді. Бірмәнді шешілу шарты бұзылған жағдайда, яғни $\triangle = 0$ болғанда, шеттік есептің шешуі жалғыз еместігі нақтыланды. Сонымен қоса Миттаг-Леффлер жалпыланған функцияларын Райт жалпыланған функциялары арқылы жазуды қолданып, $\lambda$ үлкен мәндерінде $\triangle$ функцияларының қасиеттері, яғни $\lambda \to \infty$ және $\lambda \to -\infty$ болғанда, оқылды. Райттың жалпыланған функциялары үшін асимптотикалық формулаларын қолданып, Штурм-Лиувилль типті шарттарымен берілген шеттік есептің меншікті мәндерінің сандарының ақырлылығы жайлы теорема анықталды.

*Кілт сөздер:* бөлшек ретті дифференциалдық теңдеулер, кешікпелі аргументті дифференциалдық теңдеулер, Грин функциясы, Миттаг-Леффлер жалпыланған функциясы, Райт жалпыланған функциясы.

М.Г. Мажгихова

## Метод функции Грина для дифференциального уравнения дробного порядка с запаздывающим аргументом

В статье исследована краевая задача с условиями типа Штурма-Лиувилля для линейного обыкновенного дифференциального уравнения дробного порядка с запаздывающим аргументом с постоянными коэффициентами. Условие однозначной разрешимости поставленной задачи получено в виде $\triangle \neq 0$. Для решения исследуемой задачи авторами применен метод функции Грина, в терминах которой и выписано решение краевой задачи. Функции Грина, в свою очередь, записаны в терминах обобщенной функции Миттаг-Леффлера. Доказана теорема существования и единственности решения исследуемой задачи. Отмечено, что в случае, когда условие однозначной разрешимости нарушается, то есть при $\triangle = 0$, решение краевой задачи не единственно. Используя запись обобщенной функции Миттаг-Леффлера через обобщенную функцию Райта, изучены также свойства функции $\triangle$ при больших значениях $\lambda$, то есть при $\lambda \to \infty$ и $\lambda \to -\infty$. Применяя асимптотические формулы для обобщенной функции Райта, определена теорема о конечности числа собственных значений краевой задачи с условиями типа Штурма-Лиувилля.

*Ключевые слова:* дифференциальное уравнение дробного порядка, дифференциальное уравнение с запаздывающим аргументом, функция Грина, обобщенная функция Миттаг-Леффлера, обобщенная функция Райта.

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# Closure of special atomic subsets of semantic model

The present paper concerns some properties of the so-called small models, generally speaking, not necessarily complete theories and their relationship with each other. In the well-known paper [1], R. Vaught have proved the fundamental theorem-criterion on the behavior of countable prime and atomic models for complete theories in countable language. The essence of this criterionis that in a complete theory any countable prime model is at the same time an atomic model of this theory. The result obtained in this paper is related to the classical problem of Vaught about countably prime models of complete theories but in more general formulation of the notion of countable atomicity. The main result of this paper is that it focuses on the syntactic properties on special subsets of a fragment of the semantic model the specific Jonsson theory. The concept of the so-called model-theoretic «rheostat» was also used to obtain results related to the refinement of the concept of atomicity in the framework of Jonsson's theories.

*Keywords*: Jonsson theory, semantic model, existentially prime model, atomic model, core model.

In [1] it was proved that the atomic model is countable prime. In model theory, small models are countable models with additional conditions, in particular, prime or atomic ones. Moreover, if a theory has a countable atomic model, then it is unique up to isomorphism. The atomicity criterion for the models obtained by Vaught was proved in the framework of the study of complete theories. Thus, if we are dealing with complete theories, then any theory that has a simple model has a good syntactic characteristic, namely, any element of this model implements some basic type. We will have a different situation if we investigate a more general picture: we omit the condition of completeness. In this case, instead of a prime model, the concept of an algebraically prime model is usually considered. Much has been done in this direction in [2]. But, as the results of this work showed, the criterion of small models in the framework of the study of the concepts of atomicity or algebraic primarily in [2] was not obtained. Moreover, for all kinds of atomic models that were considered in this work, examples of those theories that did not even have an algebraically prime model were built.

In connection with this dissonance between atomicity and algebraic primarily, we continue to search for additional conditions which will make it possible to find an analog of the main result from [1] for the corresponding primarily of simplicity and atomicity for models of theories considered.

In this article, we will focus our attention on the study of special models for certain types of Johnson theory within the framework of the above topics [6–8]. To have an idea of previous works concerning the behavior of small models in Johnson's theories, the following sources can be used: [4, 5]. One of the central ideas that allow us to compare the concepts of atomicity in the sense of [1] and in the sense of [2] is the idea concept of «rheostat» [4, 5]. It is clear that the larger the Johnson set, the closer the model considered to atomicity from [2] and, conversely, the smaller it is, the closer to the notion of atomicity from [1]. We fix some Johnson theory $T$ and its semantic model $C$. All sets considered in this article will be subsets of this semantic model. The fragments considered should not preserve the model-theoretical properties of the fixed Jonsson theory described above. Therefore, in each case, we will stipulate certain model-theoretical conditions under which the current problem will be considered.

Consider the necessary definitions of concepts and their properties.

*Definition 1.* The theory of $T$ is Jonson theory if:

1) the theory $T$ has infinite models;

2) the theory $T$ is inductive theory;

3) the theory $T$ has the joint embedding property ($JEP$);

4) the theory $T$ has the property of amalgam ($AP$).

The following definition of the universality and homogeneity of a model identifies the semantic invariant of any Jonsson theory, namely its semantic model. Moreover, it turned out that the saturation or unsaturation of this model significantly changes the structural properties of both Jonsson's theory and its class of models.

*Definition 2.* [8] Let $\kappa \geq \omega$. A model $M$ of theory $T$ is called a $\kappa$ -universal for $T$ if every model of $T$ with degree strictly less than $\kappa$ is isomorphically embedded into $M$; $\kappa$-homogeneous for $T$, if for any two models $A$ and $A_1$ of theory $T$, which are submodels of $M$ with the power strictly less then $\kappa$ and for isomorphism $f : A \to A_1$ for each extension $B$ of model $A$, which is a submodel of $M$ and is a model of $T$ with the power strictly less then $\kappa$ there is the extension $B_1$ of model $A_1$, which is a submodel of $M$ and an isomorphism $g : B \to B_1$ which extends $f$.

$\kappa$ is homogeneous for $T$ if, for any two models $A$ and $A_1$ of $T$, which are submodels of $M$ with power strictly less than $\kappa$, and for the isomorphism $f : A \to A_1$ for each extension $B$ of model $A$, which is a submodel of $M$ and is a model of theory $T$ with power strictly less than $\kappa$, there is an extension $B_1$ of model $A_1$, which is a submodel of $M$ and the isomorphism $g : B \to B_1$, which extends $f$.

*Definition 3.* [9] We say model $C$ of Jonsson theory $T$ is called semantic model when it is $\omega^+$-homogeneous-universal.

In accordance with the definition of the concept of Jonsson theory, it is clear that this theory is not complete. But, nevertheless, given its semantic invariant (semantic model), we can always determine the center of Jonsson's theory, which is a complete theory.

*Definition 4.* [7] The center of Jonsson's theory $T$ is called the elementary theory of its semantic model. And is denoted by $T^*$, i.e. $T^* = Th(C)$.

The «good» exclusivity of the semantic model can be judged by the following facts.

*Fact 1.* Each Jonsson theory $T$ has $k^+$-homogeneous-universal model of power $2^k$. Conversely, if the theory $T$ is inductive and has infinite model and $\omega^+$-homogeneous-universal model then the theory $T$ is a Jonsson theory.

*Fact 2.* Let $T$ be Jonsson theory then two $k$-homogeneous-universal models $M$ and $M_1$ of $T$ are elementary equivalent.

*Definition 5.* Jonsson theory $T$ is called a perfect theory if each a semantic model of theory $T$ is saturated model of $T^*$.

A result describing the perfect Jonsson theory introduced by A.R. Yeshkeyev [7].

*Theorem 1.* Let $T$ be Jonsson's theory. We conclude that the following conditions are equivalent:

1) Theory $T$ is perfect;

2) Theory $T^*$ is a model companion of theory $T$.

Let $E_T$ be the class of all existentially closed models of the Johnson theory $T$. In the general case, this class of models for an arbitrary theory may be empty. Given the well-known result of work [4], we can say that any inductive theory has a non-empty class of existentially closed models. Since Jonsson's theory is a subclass of the class of inductive theories, we can say that $E_T$ is a non-empty class.

In the case of the perfect Johnson theory, the class model center of this theory coincides with $E_T$, which follows from Theorem 2.

*Theorem 2.* If $T$ is a perfect Jonsson theory then $E_T = ModT^*$.

Let $L$ is a countable language of the first order. Let $T$ is Jonsson's theory in the language $L$ and its semantic model is $\mathcal{C}$.

We give important definitions related to the concept of the atomic set $(\nabla_1, \nabla_2)-cl$ being the central concept of this article

Let $C$ be a semantic model of some Jonsson theory $T$ in a fixed language.

*Definition 6.* [6] Model $A$ of a theory $T$ is called existentially closed if for any model $B$ and any existential formula $\varphi(\overline{x})$ with constants of $A$ we have $A \models \exists\overline{x}\varphi(\overline{x})$ provided that $A$ is a submodel of $B$ and $B \models \exists\overline{x}\varphi(\overline{x})$.

*Definition 7.* Let $A$ be a model of $T$ and $A$ can be isomorphically embedded into each model of theory $T$, then $A$ is an algebraically simple model of theory $T$.

*Definition 8.* The inductive theory $T$ is called the existentially prime if:

1) it has an algebraically prime model, the class of its $AP$ (algebraically prime models) denote by $AP_T$;

2) class $E_T$ non-trivial intersects with class $AP_T$, i.e. $AP_T \bigcap E_T \neq 0$.

*Definition 9.*A model $A$ is called atomic in meaning work [1] if every tuple of its elements satisfies some complete formula.

*Definition 10.* [2] A formula $\varphi(\overline{x})$ is a $\Delta$-formula, if exist existential formulas (from $\Sigma$) $\psi_1(\overline{x})$ and $\psi_2(\overline{x})$ such that

$$T \models (\varphi \leftrightarrow \psi_1) \quad \text{и} \quad T \models (\neg\varphi \leftrightarrow \psi_2).$$

*Definition 11.*

(i) $(A, a_0, ..., a_{n-1}) \Rightarrow_\Gamma (B, b_0, ..., b_{n-1})$ means that for every formula $\varphi(x_1, ..., x_{n-1})$ of $\Gamma$, if $A \models \varphi(\overline{a})$, then $B \models \varphi(\overline{b})$.

(ii) $(A, \overline{a}) \equiv_\Gamma (B, \overline{b})$ means that $(A, \overline{a}) \Rightarrow_\Gamma (B, \overline{b})$ and $(B, \overline{b}) \Rightarrow_\Gamma (A, \overline{a})$.

As classes $\Gamma$ we consider $\triangle$ or $\Sigma$.

Consider a complete theory $T$ in $L$. A formula $\varphi(x_1...x_n)$ is said to be complete (in $T$) iff for every formula $\psi(x_1...x_n)$ exactly one of

$$T \models \varphi \to \psi, \quad T \models \varphi \to \neg\psi$$

holds. A formula $\theta(x_1...x_n)$ is said to be completable (in $T$) if and only if there is a complete formula $\varphi(x_1...x_n)$ with $T \models \varphi \to \theta$. If $\theta(x_1...x_n)$ is not completable it is said to be incompleatable.

A theory $T$ is said to be atomic iff every formula of $L$ which is consistent with $T$ is completable in $T$. A model $A$ is said to be an atomic model iff every $n$- tuple $a_1...a_n \in A$ satisfies a complete formula in $Th(A)$

In connection with the new concept of atomicity from [2], we conclude that the following concept will be similar to the definition of the full formula.

*Definition 12.* A formula $\varphi(x_1, ..., x_n)$ is complete for $\Gamma$-formulas if $\varphi$ is consistent with $T$ and for every formula $\psi(x_1, ..., x_n)$ in $\Gamma$, having no more free variables than $\varphi$, or

$$T \models \forall\overline{x}(\varphi \to \psi).$$

Equivalently, a consistent $\varphi(\overline{x})$ is complete for $\Gamma$-formulas provided whenever as $\psi(\overline{x})$ is a $\Gamma$- formula and $(\varphi \wedge \psi)$ is consistent with $T$, then $T \models (\varphi \to \psi)$.

And the concept of the atomic model from [2] is transformed into the following concept from next the definitions

*Definition 13.* Let $B$ be $(\Gamma_1, \Gamma_2, )$ - an atomic model of $T$, if $B$ is a model of $T$ and for every $n$ every $n$-tuple of elements of $A$ satisfies some formula from $B$ in $\Gamma_1$, which is complete for $\Gamma_2$-formulas.

A generalization of the above definition is the definition of a weakly atomic model from [2].

*Definition 14.* $B$ is weak $(\Gamma_1, \Gamma_2)$ - atomic model of $T$, if $B$ is a model of $T$ and for every $n$ every $n$-tuple $\overline{a}$a of elements of $A$ satisfies in $B$ some formula $\varphi(\overline{x})$ of $\Gamma_1$ such that $T \models (\varphi \to \psi)$ as soon as $\psi(\overline{x})$ of $\Gamma_2$ and $B \models \psi(\overline{a})$.

Presenting sufficient number of examples given in [2] of this article, we will not give examples of the $(\Gamma_1, \Gamma_2)$ - atomic model and the weak $(\Gamma_1, \Gamma_2)$ atomic model, leaving the reader the opportunity to review them independently.

Next, we examine the special types of sets that we will deal with.

Let $cl$ be some closure operator that defines pregeometry over $C$ (for example, $cl = acl$ or $cl = dcl$). It is certain that such an operator is a special case of the closure operator, and its example is the closure operator, defined on any linear space as a linear hull.

Before discussing the results obtained, concerning $(\nabla_1, \nabla_2) - cl$ atomic models, we note that we fix some Jonsson theory $T$ and its semantic model $C$ in the countable language $L$ and $\nabla_1, \nabla_2 \subseteq L$ :$(\nabla_1, \nabla_2)$. Actually those sets consists of $\exists, \forall, \forall\exists$-formulas which are consistent with $T$, that is, any finite subset of formulas from $\nabla_1, \nabla_2$ are consistent with $T$. Let $A \subseteq C$.

*Definition 15.* [4] The set $A$ is called $(\nabla_1, \nabla_2) - cl$ atomic in the theory $T$, if

1) $\forall a \in A, \exists\varphi \in \nabla_1$ such that for any formula $\psi \in \nabla_2$ follows that $\varphi$ is a complete formula for $\psi$ and $C \models \varphi(a)$;

2) $cl(A) = M, M \in E_T$,

and obtained model $M$ is said to be $(\nabla_1, \nabla_2) - cl$ atomic model of theory $T$.

*Definition 16.* [4] The set $A$ is called weakly $(\nabla_1, \nabla_2) - cl$ is atomic in $T$, if

1) $\forall a \in A, \exists\varphi \in \nabla_1$ such that in $C \models \varphi(a)$ for any formula $\psi \in \nabla_2$ follow that $T \models (\varphi \to \psi)$ whenever $\psi(x)$ of $\nabla_2$ and $C \models \psi(a)$;

2) $cl(A) = M, M \in E_T$,

and obtained model $M$ is said to be weakly $(\nabla_1, \nabla_2) - cl$ atomic model of theory $T$.

It is easy to understand that definitions 18 and 19 are naturally generalized the notion of atomicity and weak atomicity to be $\nabla_1$-atom and weak $\nabla_1$-atom for any tuple of finite length from set $A$.

Let $i \in \{1, 2\}$, $M_i = cl(A_i)$, where $A_i = (\nabla_1, \nabla_2)$ is a $cl-$ atomic set. $a_0, ..., a_{n-1} \in A_1$, $b_0, ..., b_{n-1} \in A_2$.

*Definition 17.* [4]

(i) $(M_1, a_0, ..., a_{n-1}) \Rightarrow_\nabla (M_2, b_0, ..., b_{n-1})$ means that for every formula $\varphi(x_1, ..., x_{n-1})$ of $\nabla$, if $M_1 \models \varphi(\overline{a})$, then $M_2 \models \varphi(\overline{b})$.

(ii) $(M_1, \overline{a}) \equiv_\nabla (M_2, \overline{b})$ means that $(M_1, \overline{a}) \Rightarrow_\nabla (M_2, \overline{b})$ and $(M_1, \overline{b}) \Rightarrow_\nabla (M_1, \overline{a})$.

*Definition 18.* [4] A set $A$ will be called $(\nabla_1, \nabla_2) - cl$-algebraically prime in the theory $T$, if

1) If $A$ is $(\nabla_1, \nabla_2) - cl$-atomic set in $T$;

2) $cl(A) = M, M \in AP_T$,

and obtained model $M$ is said to be $(\nabla_1, \nabla_2) - cl$ algebraically prime model of theory $T$.

From the definition of an algebraically prime set in the theory $T$ follows that the Jonsson theory $T$ which has an algebraically prime set is automatically existentially prime. It is easy to understand that an example of such a theory is the theory of linear spaces.

Recall that the model $A$ of theory $T$ is called core if it is isomorphically embedded in any model of a given theory and this isomorphism exactly one.

*Definition 19.* [4] The set $A$ will be called $(\nabla_1, \nabla_2) - cl$-core in the theory $T$, if

1) $A$ is $(\nabla_1, \nabla_2)$ a $cl$ - atomic set in the theory $T$;

2) $cl(A) = M$, where $M$ is a core model of theory $T$

and obtained model $M$ is said to be $(\nabla_1, \nabla_2) - cl$ core model of theory $T$.

*Definition 20.* [5] (a) A - $(\nabla_1, \nabla_2) - cl$-atomic set in theory $T$ is said to be A - $(\nabla_1, \nabla_2) - cl$-$\Sigma$-nice-set in theory $T$, $\forall A' : A'$ - $(\nabla_1, \nabla_2) - cl$-atomic set in theory $T$, if

1) $cl(A) = M \in E_T \cap AP_T$,

and obtained model $M$ is said to be $(\nabla_1, \nabla_2) - cl$-$\Sigma$-nice model of theory $T$.

2) for all $a_0, ..., a_{n-1} \in A$, $b_0, ..., b_{n-1} \in A'$, if $(M, a_0, ..., a_{n-1}) \Rightarrow_\exists (M', b_0, ..., b_{n-1})$, then $\forall a_n \in A$, $\exists b_n \in A'$ such that $(M, a_0, ..., a_n) \Rightarrow_\exists (M', b_0, ..., b_n)$, where $M' = cl(A')$.

(b) $A$ - $(\nabla_1, \nabla_2) - cl - \Sigma^*$-nice-set in theory $T$ if the condition in (a) holds with '$\Rightarrow_\exists$' replaced both places it occurs by '$\equiv_\exists$' and obtained model $M$ is said to be $(\nabla_1, \nabla_2) - cl$-$\Sigma^*$-nice model of theory $T$.

(c) $A$ - $(\nabla_1, \nabla_2) - cl - \Delta$-nice set in theory $T$ if the condition in (a) holds with '$\Rightarrow_\Delta$' replaced both places it occurs by '$\equiv_\Delta$', where $\Delta \subseteq L$, $\Delta = \forall \cap \exists$.

and obtained model $M$ is said to be $(\nabla_1, \nabla_2) - cl - \Delta$-nice model of theory $T$.

*Theorem 3.* [5] Let $T$ be complete for $\exists$-sentences a strongly convex Jonsson perfect theory and let $M$ is $(\nabla_1, \nabla_2) - cl$-atomic model in $T$.

(a) Then $(i) \Rightarrow (ii) \Rightarrow (iii)$ and $(ii) \Rightarrow (ii)^*$ where:

$(i)$ $M$ is $(\Sigma, \Sigma) - cl$-atomic model in theory $T$,

$(ii)$ $M$ is $(\nabla_1, \nabla_2) - cl$-$\Sigma^*$-nice-model in theory $T$,

$(ii)^*$ $M$ is e.c. and $(\nabla_1, \nabla_2) - cl$-$\Sigma$-nice model in theory $T$,

$(iii)$ $M$ is weak $(\Sigma, \Pi) - cl$-atomic model in theory $T$,

(b) If $T$ is complete for $\forall\exists$ sentences, then $(i)$, $(ii)$, $(ii)^*$ and $(iii)$ are all equivalent.

*Principle of «rheostat».* [5]

Let two countable models $A_1, A_2$ of some Jonsson theory $T$ be given. Moreover, $A_1$ is an atomic model in the sense of [1], and $X$ is $(\nabla_1, \nabla_2) - cl$ -algebraically prime set of theory $T$ and $cl(X) = A_2$. Since $\nabla_1 = \nabla_2 = L$, then $A_1 \cong A_2$.

By the definition of $(\nabla_1, \nabla_2)$ - algebraic primeness of the set $X$, the model $A_2$ is both existentially closed and algebraically prime. Thus, the model $A_2$ is isomorphically embedded in the model $A_1$. Since by condition the model $A_1$ is countably atomic, then according to the Vaught's theorem, $A_1$ is prime, i.e. it is elementarily embedded in the model $A_2$. Thus, the models $A_1, A_2$ differ from each other only by the interior of the set $X$. This follows from the fact that any element of $a \in A_2 \backslash X$ implements some main type, since $a \in cl(X)$. That is, all countable atomic models in the sense of [1] are isomorphic to each other, then by increasing $X$ we find more elements that do not realize the main type and, accordingly, $cl(X)$ is not an atomic model in the sense of [1]. Thus, the principle of rheostat is that, by increasing the power of the set $X$, we move away from the notion of atomicity in the sense of [1] and on the contrary, decreasing the power of the set $X$ we move away from the notion of atomicity in the sense of [2].

Let $APC \in$ {atomic, algebraically prime, core}. Thus, by specifying the set $X$ as $(\nabla_1, \nabla_2) - cl - APC$, (where $APC$ is a semantic property), we can also specify atomicity in the sense [2] in relation to atomicity in the sense of [1]. And accordingly, according to the principle of «rheostat» after the $APC$ property is defined, we obtain the corresponding concepts of atomic models, the role of which is played $A_2$ from the principle of «rheostat».

Let us consider some properties of the types of models described above and their connection with some properties concerning the syntactic characteristics of a certain «atomicity» of existential formulas. We introduce some properties from [2], denoted by $(R_0)$ and $(R_1)$, the essence of these properties is as follows:

$(R_0)$: Every existential formula complete for $\Delta$ formulas is complete for existential formulas.

$(R_1)$: Every existential formula $\varphi(\overline{x})$ consistent with $T$ is implied by some $\Delta$ formula $\theta(\overline{x}))$ consistent with $T$.

We will say that Jonsson's theory admits $(R_0)$ and $(R_1)$ if these conditions are satisfied for all the corresponding forms of formulas compatible with the theory $T$ .

The following results characterize properties of $(R_0)$ and $(R_1)$. To prove Theorem 4, we need auxiliary lemmas. Let us give them

We will call $X$ a subset of the semantic model $C$ of the above fixed Jonsson theory $T$. Then the fragment $F$ of set $X$ is the set of all universally existential sentences true in the definable closure of this set $X$. That is, $F = Th_{\forall\exists}(cl(X))$, where $cl = dcl$ $dcl(X) = M$ $M \in ModE_T$, as well as $M$ satisfies the same conditions as the set $X$.

*Lemma 1.* Let $F$ be some fragment $(\nabla_1, \nabla_2) - cl - \Delta$-nice a.p. set of $X$. $F$ - is a perfect existentially prime complete theory for $\exists$-sentences. Then fragment $F$ entails property $(R_1)$

By of perfectness, all formulas with respect to the $F^*$ - center of the theory $F$, due to the model completeness of $F^*$ and the existential simplicity of the theory $F$, we can assume that it suffices to consider the case when $(\nabla_1 = \nabla_2)$ and equals $(\nabla)$.

*Proof.* Let $A$ be a $(\nabla_1, \nabla_2) - cl - \Delta$-nice a.p. model of theory $F$ and let $\varphi(\overline{x})$ be an existential formula consistent with $F$. Then $A \models \varphi(\overline{a})$ for some $(\overline{a})$ in $A$. Let $\{\theta_i(\overline{x}) : i \in \omega\}$ be the set of all $\Delta$ satisfied by $(\overline{a})$ in $A$. By $(\nabla_1, \nabla_2) - \Delta - nicenessa.p$, if $B$ is a model of $F$ and $B \models \theta(\overline{b})$ for all $i \in \omega$ , then $(A, (\overline{a}) \equiv_\Delta (B, (\overline{b}))$ and so $A$ can be embedded in $B$ with each $a_i$ mapped to $b_i$. Hence $B \models \varphi(\overline{b})$since $\varphi$ is existential. Therefore, $\varphi$ follows from $\theta_i : i \in \omega$ on models of $F$. By compactness, we get a single $\Delta$ formula $\theta(\overline{x})$ satisfied by $(\overline{a})$ and such that $F \models (\theta \to \varphi(\overline{a}))$, and so this fragment $F$ admits property $R_1$.

*Lemma 2.* Let $F$ be some fragment $(\nabla_1, \nabla_2)$ $- cl - \Delta - nice$ a.p. set of $X$. $F$ - is a perfect existentially prime complete theory for $\exists$-sentences. Then $R_1$ entails $R_0$.

*Proof.* By virtue of Lemma1 $F$ admits the $R_1$ property. By virtue of the perfection of the theory $F$ and the model completeness of $F^*$ without loss of generality, we can assume that there exists a $\Delta$ formula $\theta(\overline{x})$ compatible with $F$ such that $F \models \theta \to (\varphi \wedge \psi)$, where $(\varphi \wedge \psi)$ is shared with $F$. Then $F \models (\varphi \to \theta)$ since $\varphi$ is complete for $\Delta$ formulas and $\varphi \wedge \theta$ is consistent with $F$. Hence $F \models (\varphi \to \psi)$.

Therefore $\varphi$ is complete for existential formulas. Thus, $R_0$ is satisfied.

We proceed directly to the proof of the following theorem.

*Theorem 4.* Let $F$ be some fragment $(\nabla_1,\ \nabla_2) - cl - \Delta - nice$ algebraically prime set $X$ and let $A \in AP_F \cap E_F$ from the fragment $F$ is a perfect existentially prime theory, complete for $\exists$ - sentences. Then $A$ is $(\nabla_1,\ \nabla_2) - \Delta - nice$ algebraically prime set if and only if $A$ is $(\nabla_1,\ \nabla_2) - cl\ - \Delta\ -\ nice -$ atomic.

Let $F$ be some fragment $(\nabla_1, \nabla_2) - cl - \Delta - nice$ algebraically prime set $X$ and let $A \in AP_F \cap E_F$ and $F$ fragment is a complete existentially simple theory, complete for $\exists$ - sentences. Then $A$ is $(\nabla_1,\ \nabla_2) - \Delta - nice$ algebraically prime if and only if $A$ is $(\nabla_1,\ \nabla_2) - cl - \Delta$ - nice - atomic.

Let $T$ be an $\forall\exists$ theory complete for existential sentences, and let $A$ be a countable model of $T$. Then $A$ is $\triangle$-nice if and only if $A$ is $(\triangle,\ \Sigma)$-atomic.

*Proof* Suppose $A$ is $(\nabla_1,\ \nabla_2) - cl\ - \Delta\ - nice -$ atomic. Then, from the same proof of an isomorphism of the corresponding countable models by Theorem 4[5] and Theorem 3 (the proof of which can be found in the work of "Core Jonsson theories"in this volume) it follows that $A$ is $(\nabla_1,\ \nabla_2) - cl\ - \Delta - nice$. The rest follows from the perfection of the fragment $F$ and the model completeness of $F^*$.

Next, we prove in the opposite direction. Suppose $A$ is $(\nabla_1, \nabla_2) - cl\ - \Delta - nice$. Then $(R_1)$ holds by Lemma 1 and also $(R_0)$ by Lemma 2. Since $A$ is in a particularly algebraically prime and existentially closed model of $F$, we know that $F$ has $(\nabla_1, \nabla_2) - cl\ - \Delta - nice-$ atomic model.

Therefore by Theorem 4 [5] every existential formula $\psi(\overline{x})$ consistent with $F$ is implied by an existential formula $\varphi(\overline{x})$ complete for $(\Delta)$-formulas. By $(R_0)$ is, in fact, complete for existential formulas. By $(R_1)$ there is a $(\Delta)$ formula $\theta(\overline{x})$ consistent with $F$ such that $F \models (\theta \to \varphi)$. Then $\theta$ is also complete for existential formulas and $F \models (\theta \to \psi)$. So by Theorem 4 [5] $T$ has a $(\nabla_1, \nabla_2) - cl\ - \Delta - nice-$ atomic model. By Theorem 2[4] $F$ can have only one a.p. model, so the given a.p. model $A$ must be $(\nabla_1, \nabla_2) - cl\ - \Delta - nice-$-atomic.

*Theorem 5.* Let $F$ be a convex perfect existentially simple complete fragment for $\exists$ sentences of some $(\nabla_1, \nabla_2) - cl\ - \Delta - nice-$ set $X$ Then the following are equivalent:

(i) $F^*$ has a core model;

(ii) whenever $\psi(x)$ is existential and $F \models \exists x\psi$, then there is some existential $\varphi(x)$ and integer $k$ such that

$$F \models (\exists^{=k})x\varphi \wedge \exists(x)(\varphi \wedge \psi),$$

and (#) if $F \models (\sigma_1 \vee \sigma_2)$ where $\sigma_1, \sigma_2$ are existential sentences, then $F \models \sigma_1$ or $F \models \sigma_2$).

*Proof.* Since $F$ is perfect and convex, it has a unique core model that will be the core model and its center. Therefore, from (1) in (2) follows from Theorem 3 (the proof of which can be found in the work of «Core Jonsson theories» in this volume).

The proof in the opposite direction follows from the fact that Let $F^*$ be the center of $F$, then the Kaiser shell $F$ coincides with $F^*$ by virtue of perfection, and the Kaiser shell is $F^0 = Th_{\forall\exists}D$, where $D$ is the semantic model of the fragment $F$. $F^0$ has a model $M$, each element that satisfies one of the formulas $\varphi(x)$ of data by condition (2). Due to the convexity of $F$, this $M$ model is a nuclear model of the $F$ fragment. Further, since $F$ is a perfect and existentially simple theory $M$ is an existentially closed model of the center $F^*$ and by virtue of convexity, it is unique.

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## Семантикалық модельдің арнайы атомдық анықталған ішкі жиындарының тұйықтамасы

Мақалада жалпы айтқанда, теориялардың, яғни толық теория болуы міндетті емес, кішігірім модельдерінің нақты қасиеттері және олардың өзара байланысы қарастырылған. Саналымды тілдегі атомдық модельдер саналымды жай модельдер үшін фундаменталды теореманың белгілі критерий тәртібін Р. Воот [1] жұмысында дәлелдеді. Бұл критерийдің мағынасы, яғни толық теорияның кез келген моделі осы теорияның саналымды жәй саны біруақытта атомдық моделі болып табылады. Осы жұмыстағы алынған негізгі нәтиже негұрлым жалпы есеп қойылымы тұрғысынан қарағанда толық теориялардың саналымды-жай модельдер үшін Вооттың классикалық есебімен байланысты. Ұсынылып отырған жұмыстың мақсаты қандай да бір йонсондық теориялардың семантикалық моделінің фрагментінің арнайы ішкі жиындарының синтаксистік қасиеттеріне бағытталған. Сонымен қатар, йонсондық теория аясында атомдық ұғымдарын нақтылауға қатысты нәтижелерді алу үшін модельді-теоретикалық реостат ұғымы қолданылады.

*Кілт сөздер*: йонсондық теория, семантикалық модель, экзистенциалды жай модель, атомдық модель, ядролық модель.

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# Замыкание специальных атомных подмножеств семантической модели

Статья касается некоторых свойств так называемых малых моделей, вообще говоря, не обязательно полных теорий и их взаимосвязи друг с другом. В работе [1] Р. Воот доказал фундаментальную теорему-критерий поведения счетных простых и атомных моделей для полных теорий на счетном языке. Суть критерия заключается в том, что в полной теории любая модель счетного простого числа является одновременно атомной. Результат, полученный авторами статьи, связан с классической проблемой Воота о счетно простых моделях полных теорий, но в более общей формулировке понятия счетной атомности. Главным моментом этой статьи является то, что она фокусируется на синтаксических свойствах специальных подмножеств фрагмента семантической модели конкретной теории Йонсона. Концепция так называемого теоретико-модельного «реостата» была также использована для получения результатов, связанных с уточнением концепции атомности в рамках теорий Йонсона.

*Ключевые слова*: йонсоновская теория, семантическая модель, экзистенциально простая модель, атомная модель, ядерная модель.

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# Core Jonsson theories

The article concerns the description the new concept as core of Jonsson theories, also their combinations, which admit a core model in the class of existentially closed models of this theory. Along with core the property of an existentially algebraically prime theory is considered as an additional property to core Jonsson theory. This article also discusses some combinations of Johnson's theories, where the authors tried to transfer some results from [1] to Johnson's theories that satisfy the definition of core or $EAP$, or their combinations. From the definition of the core and the existentially algebraic primeness of Johnson theory, it can be noted that the core model from [1] in the framework of the study of any Johnson theory will be a unique and rigidly embedded model of the theory were considered. And thus, such a solution to the problem with respect to core models is considered for the first time.

*Keywords*: convex theory, strongly convex theory, center of Jonsson theory, semantic model, algebraically prime model, core model, core theory.

This article in its content refers to those issues of model theory that are related to the themes defined by A. Robinson and are related to the study of the theory's convexity. On the one hand, a full description of a concept of convexity is given in [1] . In work [1], a close relationship between the concepts of the theory's convexity and the concept of core model of this theory is studied. The concepts were considered in [1] are defined for arbitrary theories that, generally speaking, are not complete. Also, we note that, in the proofs of the main results of this article, the considered formulas in their complexity have a prenix length of no more than two.

In the present article, we will restrict the studied klass, generally speaking, of incomplete theories to the class of Jonsson theories. On the other hand, we will consider the concept of core models in a more general context, namely in the class of existentially closed models of the considered Jonsson theory. Due to the inductance of the Jonsson theory, its class of existentially closed models of theory is always not empty. In [1], the core structures (a model of the signature of this theory) are actually considered , not the models of theory, and as its special case, the concept of a rigidly embedded model was considered. In our case, we will consider the core model and, by definition, this concept will coincide with the concept of a rigidly embedded model, as in [1]. Accordingly, all properties of the above models of theory will be translated into the concept of the core model in our sense. The purpose of this work is to relate the results of the study of Jonsson theories [3, 4] with the study of model-theoretical properties of core. Jonsson theories that admit a core model in the class of existentially closed models of this theory will be called the core Jonsson theories. It is clear that any core model is an algebraically prime model [2]. In [3, 4], new types of atomic and prime countable models of the corresponding types of Jonsson theories were considered.

To obtain the main results of this article, we give the necessary definitions of concepts and their model-theoretical properties. For more in-depth information on Jonsson theories, please refer to the following sources [3, 4]. Nevertheless, we give some basic definitions and related results.

Consider the following definitions:

*Definition 1.* A theory $T$ is called a Jonsson theory if:

1) the theory $T$ has infinite models;
2) the theory $T$ is inductive;
3) the theory $T$ has the joint embedding property ($JEP$);
4) the theory $T$ has the property of amalgam ($AP$).

Examples of Jonsson theories are:

1) the group theory,
2) the theory of Abelian groups;
3) the theory of fields of fixed characteristics;
4) the theory of Boolean algebras;

5) the theory of polygons over a fixed monoid;
6) the theory of modules over a fixed ring;
7) the theory of linear order.

When studying the model-theoretic properties of Jonsson theory, the semantic method plays an important role, i.e. the elementary properties of the center of Jonsson theory are in a certain sense associated with the corresponding first-order properties of Jonsson theory itself. The center of Jonsson theory is a syntactic invariant and its properties are well defined in the case when Jonsson theory is perfect. The following concepts define the essence of the semantic model and the center of Jonsson theory [6].

*Definition 2.* Let $\kappa \geq \omega$. Model $M$ of theory $T$ is called $\kappa$-universal for $T$, if each model $T$ with the power strictly less $\kappa$ isomorphically imbedded in $M$; $\kappa$- homogeneous for $T$, if for any two models $A$ and $A_1$ of theory $T$, which are submodels of $M$ with the power strictly less then $\kappa$ and for isomorphism $f : A \to A_1$ for each extension $B$ of model $A$, wich is a submodel of $M$ and is model of $T$ with the power strictly less then $\kappa$ there is exist the extension $B_1$ of model $A_1$, which is a submodel of $M$ and an isomorphism $g : B \to B_1$ which extends $f$.

*Definition 3.* A model $C$ of a Jonsson theory $T$ is called semantic model, if it is $\omega^+$-homogeneous-universal.

*Definition 4.* The center of a Jonsson theory $T$ is an elementary theory $T*$ of the semantic model $C$ of $T$, i.e. $T^* = Th(C)$ [8].

*Fact 1 [6].* Each Jonsson theory $T$ has $k^+$-homogeneous-universal model of power $2^k$. Conversely, if a theory $T$ is inductive and has infinite model and $\omega^+$-homogeneous-universal model then the theory $T$ is a Jonsson theory.

*Fact 2 [6].* Let $T$ is a Jonsson theory. Two $k$-homogeneous-universal models $M$ and $M_1$ of $T$ are elementary equivalent.

*Definition 5.* A model $A$ of theory $T$ is called existentially closed if for any model $B$ and any existential formula $\varphi(\overline{x})$ with constants of $A$ we have $A \models \exists\overline{x}\varphi(\overline{x})$ provided that $A$ is a submodel of $B$ and $B \models \exists\overline{x}\varphi(\overline{x})$.

We denote by $E_T$ the class of all existentially closed models of the theory $T$.

In connection with this definition in the frame of the study of inductive theories, the following two remarks are true:

*Remark 1:* For any inductive theory $E_T$ is not empty.

*Remark 2:* Any countable model of the inductive theory is isomorphically embedded in some countable existentially closed model of this theory.

An analogue of a prime model (in the sense of a complete theory) for an inductive model, generally speaking, incomplete theory, is the concept of an algebraically prime model, which introduced A. Robinson [5].

*Theorem 1 [9].* Let $T$ be a Jonsson theory. Then the following conditions are equivalent:
1) the theory $T$ is perfect;
2) the theory $T^*$ is a model companion of $T$.

*Theorem 2 [10].* If $T$ is a perfect Jonsson theory then $E_T = ModT^*$.

When studying the model-theoretic properties of an inductive theory, so called existentially closed models play an important role. Recall their definitions.

*Definition 6.* A model of theory is called an algebraically prime, if it is isomorphically embedded in each model of the considered theory.

Note that since the class of Jonsson theories of a fixed signature is a subclass of inductive theories of this signature, then the above remarks 1,2 are true for Jonssons theories and, by criterion of Jonsson theory's perfectness, class of existentially closed models of considered Jonsson theory coincides with the class of center's model of this theory.

In connection with the interest to the $AAP$ problem in the frame of the study of Jonsson theory in [1] a new class of theories was defined, in which there is an algebraically prime model which is existentially closed.

Recall the definition of this class.

*Definition 7.* A theory  is called **convex** if for any its model $A$ and for any family $\{B_i \mid i \in I\}$ of substructures of $A$, which are models of the theory $T$, the intersection $\bigcap_{i\in I} B_i$ is a model of $T$, provided it is non-empty. If in addition such an itnersection is never empty, then $T$ is called **strongly convex**.

*Definition 8.* The model $A$ of theory $T$ is called core if it is isomorphically embedded in any model of a given theory and this isomorphism exactly one.

Recall the definition of a rigidly embedded model from [1].

*Definition 9.* The model $A$ of theory $T$ is rigidly embeddable in model $B$ of theory $T$, if there is exactly one isomorphism of $A$ into $B$. It is clear that $A$ is rigidly embeddable in every model of $T$ if and only if $A$ is a core model for $T$ and has no proper automorphisms except identical. Thus, any core model of the core Jonsson theory is rigidly embeddable in any existentially closed model of this theory.

All the new definitions of theories given below distinguish a fairly wide natural subclass of theories among the class of inductive theories. The relevance of studying this class of theories is expressed by the fact that each of the above classes of theories is determined by a natural concept that generalizes the well-known concepts of core, algebraic simplicity, and their combinations. At the same time, new classes of theories become interesting for studying their model-theoretical properties of incomplete theories as part of the study of Johnson theories that are associated with the above concepts.

Since, we will deal with, generally speaking, incomplete Jonsson theories and their classes of existentially closed models in connection with the study of core models, we distinguish a natural subclass of the class of all Jonsson theories, which is naturally connected with the concept of a core model. We give the following definition.

*Definition 10.* An inductive theory $T$ is called a core theory if there exists a model $A \in E_T$ such that for any model $B \in E_T$ there exists a unique isomorphism from $A$ to $B$.

Since, by definition, any core model is an algebraically prime model, we distinguish a natural subclass of all Jonsson theories' class, namely, the class of such Jonsson theories that necessarily have an algebraically prime model. We give the following definition.

*Definition 11.* Theory $T$ is called existentially algebraically prime ($EAP$) if it has a model $A \in E_T$ such that for any $B \in E_T$, $A$ is isomorphically embedded in $B$.

*Definition 12.* The inductive theory $T$ is called the existentially prime if:

1) it has a algebraically prime model, the class of its $AP$ (algebraically prime models) denote by $AP_T$;

2) class $E_T$ non trivial intersects with class $AP_T$, i.e. $AP_T \bigcap E_T \neq 0$.

On the other hand, in [1] the notion of core model was studied in the framework of the study of convex theory or strongly convex theory. Therefore, later in this article we will consider the property of convex and strongly convex of the considering theory, as an additional concept to the core Jonsson theory.

The following fact about the realization of existential formulas with respect to extensions is well known.

*Lemma 1.* Assume that $A \subseteq B$ are models of $\exists^{=n} x\phi$, where $\phi(x)$ is existential formula. Then

$$\{a \in A : A \models \phi[a]\} = \{b \in B : B \models \phi[b]\}.$$

*Proof* follows from the fact that existential formulas are closed with respect to extensions.

Sometimes, we will need structures with special properties, and we will deal with theories that satisfy certain model-theoretical conditions. In the remainder of this section, we determine the properties that we will use and state some elementary facts concerning them. For more information, see [7] and [5].

To denote that $B$ satisfies every true sentence of an existential sentence on $A$ we write $A \exists B$ . $Th(C)$, the complete theory of $C$, is the set of all sentences true on $C$.

The convex theories have an important algebraic property: let $T$ be a convex theory, then for any model $A$ of $T$, any nonempty subset $B \subseteq A$ generates a single substructure, which is a model of the $T$. In particular, the intersection of all models of $T$ contained in this model and which contain this set $B$. If the theory of $T$ is strongly convex, then the intersection of all models of $T$ contained in this model of $T$ is also a model of $T$. This intersection is called the core model of $T$. In [1] noted that if $T$ satisfies a joint embedding property and it is strongly convex, then the core model of this theory is unique up to isomorphism.

In the remainder of this article, we will deal with the above mentioned combinations of Jonsson theories. Hence, we will try to transfer some results from [1] to Jonsson theories that satisfy Definition 10 or Definition 11, or their combinations. What is the meaning here? The fact is that from the definition of core and the existentially algebraically primeness of Jonsson theory, it can be noted that a core model from [1] in the framework of the study of any Jonsson theory will be unique and rigidly embeddable model of the considering theory. And thus, such statement of the problem regarding of the core models is considered for the first time.

*Theorem 3.* For any core perfect Jonsson theory $T$, the following conditions are equivalent:

1. $A$ is core model of $T$
2. $A$ is rigidly embeddable in any existentially closed of model $T$
3. $A$ is a model of center of $T^*$ and exist an existential formulas $\phi_i(x)$ and $k_i \in \omega$ for $i \in I$ exist , such that

$$A,\, T^* \models \exists^{=k_i} x\phi_i \; \forall i \in I,$$

and

$$A \models \forall x \bigvee_{i \in I} \phi_i$$

*Proof.*

The equivalence of items (1) and (2) follows from the fact that the theory is core and perfect. Let us prove from (3) to (2) Let $B$ be some model of the theory $T^*$, then there exists $B'$ such that $B'$ is an existentially closed model of $T^*$, where $B$ is elementary embedded in $B'$ relative to existential formulas. Such $B'$ exists due to the inductance of $T^*$. Moreover, the power of $B'$ can be any power that less or equal to the power of the semantic model of this theory. It is clear that $A$ is elementary embedded in $B'$ with respect to existential formulas, then in $B'$ there is an existentially closed submodel $A'$ such that $A$ is isomorphic to $A'$.

$$A' = \{a \in A' : A' \models \bigvee_{i \in I} \phi_i[a]\}$$

where $\phi_i$ are existential formulas such that $A'$, $B'$ and $B$ are models of $\exists^{=k_i} x\phi_i$. By Lemma 1, therefore

$$A' = \{b \in B : B' \models \bigvee_{i \in I} \phi_i[b]\} = \{b \in B : B \models \bigvee_{i \in I} \phi_i[b]\}$$

Hence $A' \subseteq B$ and therefore $A$ is the core model of $T$.

(2) $\to$ (3)

By virtue of the perfectness of theory follows that $A$ is a model of the center and, due to of its coreness, it is embedded exactly once in any model of this center. Further, due to the fact that the center is a model companion of $T$ (since the theory $T$ is perfect), and the model companion is a model-complete theory. Accordingly, in a model-complete theory any formula is equivalent to some existential formula. This implies condition (3)

*Corollary 1.* $C$ is the core model of some perfect core Jonsson theory $T$ if and only if $C$ is the core model of the Kaiser Hull $T^0$ of $T$.

*Proof:* Let us prove the necessity of the statement.

Let $C$ be the core model of the above perfect core Jonsson theory $T$.

Let $M$ be semantic model of $T$. Let $T^0$ be the Kaiser hull of $T$, i.e.

$$T^0 = \{\varphi \in L_0 \ : \varphi \in \forall\exists \, sentences \, and \, M \models \varphi\}$$

where $\varphi$ are the set of all sentences of the signature language of the theory T.

Such that $T$ is perfect, then $T^*$ is a model companion of $T$ and hence, a model complete theory. As a consequence of this, any formula in $T^*$ is equivalent to some existential formula. Since $M$ is a semantic model of $T$ and a model of $T^*$, then the Kaiser Hull $T^0$ will be equal to the center of the theory $T$, i.e. $T^*$, where

$$T^* = \{\varphi \in L_0 \text{ (the set of all sentences of the signature's language of T)} : \, in \, M \models \varphi\}.$$

The condition (3) of Theorem 3 holds for the model $C$ and $T^*$, and it follows that $C$ is a core model of the theory $T^*$. Let $A$ be an arbitrary model of $T^*$, then $C$ is isomorphically embedded in $A$ in a unique way, that is, there is $C'$ such that $C' \subseteq A$, where $C'$ is isomorphic to $C$. Moreover, $C'$ is isomorphically embedded in every model of $T^*$.

$$C' = \cap\{B : B \subseteq A \, and \, B \models T^*\}$$

Thus, it turns out that $T^*$ is strongly convex and that $C$ is the core model of theory $T^*$. $\sharp$.

Let us prove the sufficiency of the condition. Suppose that $T^*$ is a strongly convex theory and that $C$ is the core model of theory $T^*$. Let $A$ be a model of $T$ and let $M$ be a semantic model of theory $T$. Then $C \in M$ and $C \cong C_1$ for some $C_1 \subseteq M$. Since $C$, which is a model of theory $T^*$, has no proper submodel we can say that there is a model.

$$C_1 = \cap\{B : B \subseteq M \, and \, B \models T\}.$$

In particulary, $C_1 \subseteq A$. If $C$ is isomorphic to some other $C_2 \subseteq A$, then it is easy to show in a similar way, that $C_1 = C_2$, and therefore $C$ is a core model of the theory $T$.

*Corollary 2.* Let $C$ be the core model of the strongly convex perfect core Jonsson theory $T$. Then there exist existential formulas $\phi_i(x)$ for $i \in I$, such that

$$T^* \models \exists^{<\omega} x\phi_i \, for \, all \, i \in I,$$

and

$$C \models \forall x \bigvee_{i \in I} \phi_i$$

*Corollary 3.* The core model $C$ is rigidly embeddable in each model of $T$ if and only if condition (3) of Theorem 3 is satisfied with $k_i = 1$ for all $i \in I$.

*Theorem 4.* Let $C$ be the core model for some existentially algebraically simple theory $T$. Then the following conditions are equivalent.

(1) $C$ is embedded in every existentialy closed model of the center of this theory.

(2) $C$ is an algebraically prime model of the theory $T$.

*Proof:* We prove from (1) to (2). Let model $C$ be embedded in each existential closed model of the center of this theory. Suppose that model $A$ does not belong to $E_T$ and suppose that model $C$ is not embedded in model $A$. Since $T$ is a Jonsson theory, then by the inductance of this theory there exists a model $B \in E_T$ such that $A$ is isomorphically embedded in $B$, but model $B$ is isomorphically embedded in the semantic model $M$ of $T$. The model $B$ also belongs to set of existentially closed models of the theory $T^*$.

Let $C'$ be an isomorphic image of the model $C$ in the model $B$. The model $A'$ is an isomorphic image of the model $A$ in the model $B$, if $C'$ is embedded in $A'$, then we get a contradiction with our assumption that $C'$ is not embedded in $A$. Therefore, suppose that $C'$ is not embedded in $A'$ and they are not isomorphic. From this it follows that there is a formula that distinguishes them. Let this formula be $\varphi(x)$. Without loss of generality, suppose that in $C' \models \varphi[c]$ where $c \in C'$, that is, in $C' \models \exists x\varphi(x)$ but by assumption in the model $A'$, which is isomorphic to the model $A$, it will be true that $A' \models \neg\varphi[a]$ where $a \in A'$. But both $A'$ and $C'$ belong to the model $B$, which is existentially closed in accordance with the above. Then $B \models \exists x(\varphi(x)\&\neg\varphi(x))$ we get a contradiction. So the model $C$ is an algebraically prime model of the theory $T$

From 2 we prove 1. Let $C$ be an algebraically prime model of the theory $T$. Thus $C$ is isomorphically embedded into any model $B \in ModT$, but since $T \subseteq T^*$ we have $ModT^* \subseteq ModT$. It follows that the model $C$ is algebraically prime for the theory $T^*$, and from this we conclude that $C$ is isomorphically embedded in any model from $E_T^*$, because $E_T \subseteq ModT^*$

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## Ядролық йонсондық теориялар

Мақаладағы негізгі нәтижелер ол йонсондық теориялардың ядролылығы сияқты жаңа ұғымды және де осы теориядағы экзистенциалды тұйық модельдер класындағы ядролы модельді рұқсат ететін комбинацияларды сипаттау болып табылады. Ядролықпен қатар осы теориядағы ядролы йонсондық теориялардың экзистенциалды алгебралық жай теориялардың қасиеттері қарастырылған. Сонымен қатар, авторлар [1] кейбір нәтижелерді ядролық немесе EAP анықтамаларын немесе олардың комбинацияларын қанағаттандыратын Йонсон теорияларына көшіріп көрді, яғни Йонсон теорияларының кейбір комбинацияларын қарастырды. Йонсон теорияларының ядролылық және экзистенциалды жай анықтамаларынан [1] ядролық модель кез келген Йонсондық теорияның зерттеу аясы тұрғысынан алсақ та, қарастырылып отырған теорияның ерекше және қатаң енгізілімді моделі болатынын атап кетуге болады. Сондықтан, ядролық модельдерге қатысты қойылған мәселе алғашқылардың бірі болып табылады.

*Кілт сөздер*: дөңес теория, қатты дөңес теория, йонсон теориясының центрі, семантикалық модель, алгебралық жай модель, ядролы модель, ядролы теория.

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## Ядерные йонсоновские теории

Основным результатом статьи является описание нового понятия как ядерность йонсоновских теорий, а также их комбинации, которые допускают ядерную модель в классе экзистенциально замкнутых моделей этой теории. Наряду с ядерностью рассмотрено свойство экзистенциально алгебраически простой теории, как дополнительное свойство ядерной йонсоновской теории. Также авторами изучены некоторые комбинации теорий Йонсона, где они попытались перенести некоторые результаты из [1] в теории Йонсона, которые удовлетворяют определению ядерности или $EAP$, или их комбинациям. Из определения ядерности и экзистенциально алгебраической простоты теории Йонсона можно отметить, что ядерная модель из [1] в рамках изучения любой теории Йонсона будет уникальной и жестко вложимой моделью рассматриваемой теории. И в заключении отметим, что такая постановка проблемы относительно ядерных моделей изучена впервые.

*Ключевые слова*: выпуклая теория, сильно выпуклая теория, центр йонсоновской теории, семантическая модель, алгебраически простая модель, ядерная модель, ядерная теория.

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# МЕХАНИКА
# MECHANICS

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## New exact particular analytical solutions of the triangular restricted three-body problem

The triangular restricted three-body problem is studied in special non-inertial central reference frame with origin at forces centre of this problem. Masses are arbitrary values. We studied the solutions of dimensionless differential equations of motion of the triangular restricted three-body problem in rotating reference frame in the pulsating variables. For the non-circular planar restricted three-body problem we have found out new exact analytical solutions. In these solutions, all the three bodies form an isosceles triangle with variable height. Also, we have found new class of analytical solutions of the planar circular restricted three-body problem in the form of non-isosceles triangle. The basis of this non-isosceles triangle is distance between the primary bodies, the ratio of sides of non-isosceles triangle is constant and infinitesimal small body is at vertex of this non-isosceles triangle. Obtained exact particular analytical solutions can be used for topological analysis of the general three-body problem.

*Keywords:* restricted three-body problem, non-inertial reference frame, invariant of center of forces, exact particular analytical solutions.

*Introduction*

We considered the restricted three-body problem with constant masses $m_1$, $m_2$, $m_3$. The condition of the restricted three-body problem statement is $m_2 \ll m_3,\ m_2 \ll m_1,\ m_3 \geq m_1$. It is widely known that at random masses of the primary bodies $m_1$ и $m_2$, the restricted three-body problem has the exact particular solutions - Lagrange solutions, when all the three bodies form an equilateral triangle [1–3]. Also there exist the solution in the form of isosceles triangle when masses of the primary bodies are equal to each other [1–3]. The problem has various applications, but the general analytical solution of this problem in finite form is not found. Due-to this, lot aspects of this problem are studied by different methods and there are plenty publications on this problem. In [4], there have been done orbit classification with numerical computation of the planar restricted three-body problem. In the work [5], good review on resonance of the Lidov-Kozai. In the work [6], there have been considered various applications of the restricted three-body problem to the Earth-Moon system and the Pluto-Charon system. The libration point orbits of the system the Earth and the Moon is described in the work [7].

In the work [8], the perturbing planar circular restricted three-body problem is used to study the restricted n-body problem. In the work [9], the elliptical restricted three-body problem is investigated and energy analysis has been conducted. In the work [10], the short-term capture of an asteroid is studied in the system Sun-Moon in the framework of the restricted four-body problem. In the work [11], based on the planar elliptical restricted three-body problem, calculation method of energy variation for one and two-impulse powered swing-by of spacecraft proposed for the Earth-Moon system. In the work [12], the charged restricted three-body problem is studied, linear stability of planar solutions is investigated and resonance curves are analyzed. In the work [13], the very long-term evolution of the hierarchical restricted three-body problem with the Lidov-Kozai cycles. In the work [14], the invariant manifold structures of the collinear libration points of the restricted three-body problem are investigated. In the work [15], through numerical simulation of the restricted elliptical three-body problem the borders of stable regions around the secondary body found. In the work [16], the existence and stability of the non-collinear libration points in the restricted three-body problem when both the primaries are ellipsoid with equal mass and identical shape are investigated. The two planet three-body problem composed of a central star and two massive planets is investigated and the authors show that secular dynamics of this system can be described using only two parameters, the ratios of the semi-major axes and the planetary masses [17].

In the paper [18], the stability of the equilibrium points under the influence of the small perturbations in the Coriolis and centrifugal forces, together with the effects of oblateness and radiation pressures of the primaries is investigated. in the work [19], the elliptic isosceles restricted three-body problem with consecutive collision is investigated and the existence of many families of periodic solutions has been proved. In the paper [20], circumbinary accretion discs in the framework of the restricted three-body problem is investigated through numerical solutions of viscous hydrodynamics equations and implicit changes of behavior of the disc near some mass ratio.

Above mentioned analysis of publications shows that the search for new exact particular analytical solutions for random masses $m_1$ and $m_3$ is important task. This work is a continuation of our research done in the paper [21]. In this work, we study analytically the triangular restricted three-body problem, when three bodies form triangle during all the time of motion. The problem is studied in the special non-inertial central reference frame with the origin at the center of forces [2, 21] through using invariant of center of forces.

## *2 Equations of motion of the restricted three-body problem in different reference frames and invariants of center of forces*

2.1. Classical equations of motion of the restricted three-body problem in absolute reference frame.

In an absolute reference frame $OX^*Y^*Z^*$ the differential equations of motion of the restricted three-body problem with constant masses $m_1$,$m_2$ and $m_3$, can be written in the following way [1–3]

$$\ddot{\vec{R}}_1^* = \vec{F}_1^* = fm_3\frac{\vec{R}_3^* - \vec{R}_1^*}{R_{13}^{*3}}, \quad \ddot{\vec{R}}_3^* = \vec{F}_3^* = fm_1\frac{\vec{R}_1^* - \vec{R}_3^*}{R_{31}^{*3}}, \tag{1}$$

$$\ddot{\vec{R}}_2^* = \vec{F}_2^* = f\left(m_1\frac{\vec{R}_1^* - \vec{R}_2^*}{R_{21}^{*3}} + m_3\frac{\vec{R}_3^* - \vec{R}_2^*}{R_{23}^{*3}}\right), \tag{2}$$

In these equations $\vec{R}_i^*$ - radius-vector, $\vec{R}_{ij}^*\,(i \neq j)$ – distances between the bodies. Differentiation in time $t$ is denoted by dot over symbol. The system of differential equations (1) describes the two-primary bodies problem with masses $m_1$, $m_3$. From this differential equations system, one can obtain the well-known relation

$$m_1\vec{R}_1^* + m_3\vec{R}_3^* = \vec{a}^*t + \vec{b}^*, \quad \vec{a}^* = \overrightarrow{const}, \quad \vec{b}^* = \overrightarrow{const}\,. \tag{3}$$

The equation of motion (2) describes motion of infinitely small body $m_2$ in the Newtonian gravity field of the two primary bodies $m_1$, $m_3$ - the classical restricted three-body problem.

2.2. Differential equations of the restricted three-body problem in the special non-inertial central reference frame and invariants of center of forces.

Then we go to the special non-inertial central reference frame through the formulas $\vec{R}_i^* = \vec{R}_G + \vec{r}_i$, $i = 1, 2, 3$. where $\vec{R}_G$ - radius-vector of the forces center $G$ in the absolute reference frame, $\vec{r}_i$- radius-vectors of the bodies in the special reference frame. The axes of the new reference frame are $Gxyz$ parallel to the corresponding axes of the absolute reference frame $OX^*Y^*Z^*$. The differential equations of the restricted three-body problem (2) in the special non-inertial central reference frame $Gxyz$ have been obtained in the work [21]

$$\ddot{\vec{r}}_2 - \vec{F}_2 = \vec{W}, \tag{4}$$

$$\vec{F}_2 = f\left(m_1\frac{\vec{r}_1 - \vec{r}_2}{\Delta_{21}^3} + m_3\frac{\vec{r}_3 - \vec{r}_2}{\Delta_{23}^3}\right), \tag{5}$$

$$\vec{W} = \vec{W}(t) = -f\frac{(m_1 - km_3)}{k+1}\frac{\vec{r}_{31}}{r_{31}^3} + 2\dot{\vec{r}}_{31}\frac{d}{dt}\left(\frac{1}{1+k}\right) + \vec{r}_{31}\frac{d^2}{dt^2}\left(\frac{1}{1+k}\right) \tag{6}$$

where the dimensionless parameter of the problem is denoted by

$$\frac{r_3}{r_1} = k = k(t) > 0, \tag{7}$$

$\Delta_{ij}$ - distances between infinitesimal small body and the primary bodies

$$\Delta_{21} = [(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]^{1/2} = \Delta_{12},$$

$$\Delta_{23} = [(x_2 - x_3)^2 + (y_2 - y_3)^2 + (z_2 - z_3)^2]^{1/2} = \Delta_{32},$$

$\vec{r}_{31}(x_{31}, y_{31}, z_{31})$ - solution of the differential equations of the two primary bodies system, which can be obtained from the (1)

$$\ddot{\vec{r}}_{31} = -f\frac{m_3 + m_1}{r_{31}^3}\vec{r}_{31}.$$

From the integral

$$\vec{r}_{31} \times \dot{\vec{r}}_{31} = \vec{c}_{31} = \overrightarrow{const} \neq 0 \tag{8}$$

one can see that the orbit is planar and the orbit is on the plane $Gxyz$. The equation (3) can be rewritten in the following form

$$(m_1 + m_3)\vec{R}_G + m_1\vec{r}_1 + m_3\vec{r}_3 = \vec{a}^*t + \vec{b}^*, \tag{9}$$

in order to define the origin of the special non-inertial reference frame, one needs to know the dimensionless variable $k$. If one obtains $k$, then taking into account (7)

$$\vec{r}_{31} = \vec{r}_1 - \vec{r}_3 = \vec{r}_1 - r_3(-\vec{e}_1) = \vec{r}_1 - (kr_1)(-\vec{e}_1) = \vec{r}_1 + k\vec{r}_1 = (1+k)\vec{r}_1\,.$$

Therefore

$$\vec{r}_1 = \frac{1}{1+k}\vec{r}_{31}, \quad \vec{r}_3 = -\frac{k}{1+k}\vec{r}_{31}.$$

Then from the equation (9), it is possible to define the origin of special non-inertial central reference frame

$$(m_1 + m_3)\vec{R}_G = \vec{a}^*t + \vec{b}^* - (m_1\vec{r}_1 + m_3\vec{r}_3) = \vec{a}^*t + \vec{b}^* - \frac{m_1 - km_3}{1+k}\vec{r}_{31}. \tag{10}$$

Thus, defining the origin of the special non-inertial central reference frame leads to the defining the parameter $k$. In accordance to the definition of special reference frame, the force$\vec{F}_2$ is directed to center of forces $G$ all the time - to the beginning of the new reference frame. That is why

$$\vec{F}_2 \times \vec{r}_2 = 0. \tag{11}$$

The equation (11) defines invariant of center of forces established in our work [21]. Invariant of forces center of the restricted three-body problem in the special non-inertial central reference frame in scalar form is

$$\left(\frac{m_3}{\Delta_{23}^3}r_3 - \frac{m_1}{\Delta_{21}^3}r_1\right)r_2\sin\alpha = 0, \tag{12}$$

where $\alpha$ - is angle between the vectors $\vec{r}_1$ and $\vec{r}_2$. Thus, in the special non-inertial central reference frame, regardless of the primary bodies masses and properties of triangle formed by three bodies, the equation (12) is right for the restricted three-body problem during all time of motion.

### *3 The triangular restricted three-body problem.*

From mathematical point of view, the equation (12) takes a place in several cases. In this work we study only one case

$$\frac{m_3}{\Delta_{23}^3}r_3 - \frac{m_1}{\Delta_{21}^3}r_1 = 0, \quad r_2\sin\alpha \neq 0. \tag{13}$$

Other cases of fulfilment of forces center invariant (12) will be considered in another works. In the case (13) all three bodies form triangle during all time of motion. Size, shape and orientation of triangle changes over time. The equation (13) takes a place in the triangular restricted three-body problem. Taking into account the equation (7), the first equation in (13) can be rewritten as

$$\left(\frac{\Delta_{23}}{\Delta_{21}}\right)^3 = k\frac{m_3}{m_1}. \tag{14}$$

Thus in the special non-inertial central reference frame, regardless from primary bodies masses and properties of triangle formed by three bodies, the equation (14) is always right for the triangular restricted three-body problem.

In vector form, the invariant of forces center in the triangular restricted three-body problem can be written

$$m_1\Delta_{23}^3\vec{r}_1 + m_3\Delta_{21}^3\vec{r}_3 = 0. \tag{15}$$

Let us consider the equations of motion of the triangular restricted three-body problem (4)–(7) in the special non-inertial central reference frame, in the general case, when $k = k\,(t) \neq const$, $\vec{r}_2 = \vec{r}_2\,(x_2, y_2, z_2)$. The invariant of forces center of the triangular restricted three-body problem (14) or (15) can be rewritten as

$$\Delta_{21} = \left(\frac{m_1}{m_3 k}\right)^{1/3}\Delta_{23}. \tag{16}$$

In our work [21], using invariant of center of forces (16) and geometrical properties of triangle, the differential equations (4)–(6) can be rewritten as

$$\ddot{x}_2 + \frac{\mu_2 x_2}{(r_2^2 + \sigma_2^2 r_{31}^2)^{3/2}} = W_x\ , \quad \ddot{y}_2 + \frac{\mu_2 y_2}{(r_2^2 + \sigma_2^2 r_{31}^2)^{3/2}} = W_y \quad , \tag{17}$$

$$\ddot{z}_2 + \frac{\mu_2 z_2}{(r_2^2 + \sigma_2^2 r_{31}^2)^{3/2}} = 0. \tag{18}$$

with the following designation

$$\mu_2 = f\frac{(m_3^{2/3} + m_1^{2/3}k^{1/3})^{3/2}}{(1+k)^{1/2}} > 0, \quad \sigma_2^2 = \frac{k}{(k+1)^2} > 0. \tag{19}$$

$$W_x = B_2\frac{x_{31}}{r_{31}^3} + D_2\dot{x}_{31} + E_2 x_{31}\ , \quad W_y = B_2\frac{y_{31}}{r_{31}^3} + D_2\dot{y}_{31} + E_2 y_{31}\ , \tag{20}$$

$$D_2 = 2\frac{d}{dt}\left(\frac{1}{1+k}\right), \quad E_2 = \frac{d^2}{dt^2}\left(\frac{1}{1+k}\right), \quad B_2 = -f\frac{m_1 - km_3}{k+1}. \tag{21}$$

The equations (17), (18), in accordance to the solution of the two-body problem, in the case (8), describes the elliptical (in particular circular), hyperbolic or parabolic triangular restricted three-body. The forces center invariant (16) can be rewritten as

$$\left(x_2 - \frac{1}{1+k}x_{31}\right)^2 + \left(y_2 - \frac{1}{1+k}y_{31}\right)^2 + \left(z_2 - \frac{1}{1+k}z_{31}\right)^2 =$$
$$= \left(\frac{m_1}{km_3}\right)^{2/3}\left[\left(x_2 + \frac{k}{1+k}x_{31}\right)^2 + \left(y_2 + \frac{k}{1+k}y_{31}\right)^2 + \left(z_2 + \frac{k}{1+k}z_{31}\right)^2\right]. \tag{22}$$

The system of equations (17)–(21) and (22) have four scalar values $x_2, y_2, z_2, k$, that is why these four scalar equations represent closed system of equations describing the triangular restricted three-body problem.

*4 The differential equations of the triangular restricted three-body problem, in the rotating special non-inertial central reference frame in the pulsating variables*

Let us consider the problem in the rotating special non-inertial central reference frame $G\xi\eta\zeta$ in the dimensionless pulsating variables. The new axe $\xi$ go through the bodies with masses $m_3$ and $m_1$. The transition formulas are following [1–3, 21]

$$x_2 = r \cdot \xi \cos\theta - r \cdot \eta \sin\theta, \quad y_2 = r \cdot \xi \sin\theta + r \cdot \eta \cos\theta, \quad z_2 = r \cdot \zeta, \tag{23}$$

$$d\theta = \frac{c}{r^2} dt, \quad r_2^2 = \xi^2 + \eta^2 + \zeta^2 = r^2 \rho^2. \tag{24}$$

In the analytical expressions (23), (24), the values $r = r(t) = r_{31}$ and $\theta = \theta(t) = \theta_{31}$ are defined by solution of the two-body problem. In the rotating special non-inertial central reference frame in dimensionless pulsating variables, the differential equations of the triangular restricted three-body problem are [21]

$$\xi'' - 2\eta' - \frac{1}{1 + e\cos\theta}\left(1 - \frac{A}{(\rho^2 + \sigma_2^2)^{3/2}}\right)\xi = \frac{1}{1 + e\cos\theta} B + s'', \tag{25}$$

$$\eta'' + 2\xi' - \frac{1}{1 + e\cos\theta}\left(1 - \frac{A}{(\rho^2 + \sigma_2^2)^{3/2}}\right)\eta = 2s', \tag{26}$$

$$\zeta'' + \frac{1}{1 + e\cos\theta}\left(e\cos\theta + \frac{A}{(\rho^2 + \sigma_2^2)^{3/2}}\right)\zeta = 0, \tag{27}$$

where dimensionless variables are

$$A = \frac{\left[1 + \nu^{2/3} k^{1/3}\right]^{3/2}}{(1+k)^{1/2}(1+\nu)} = \frac{\left(s^{1/3} + \nu^{2/3}(1-s)^{1/3}\right)^{3/2}}{1+\nu} > 0, \quad s = \frac{1}{1+k}, \tag{28}$$

$$B = \frac{k - \nu}{(k+1)(1+\nu)} = \frac{1 - s(1+\nu)}{1+\nu}, \quad \nu = \frac{m_1}{m_3} = const > 0, \quad \sigma_2^2 = s - s^2. \tag{29}$$

In the equations (25)–(27) and further, differentiation in $\theta$ is denoted by stroke. Invariant of the center of forces of the triangular restricted three-body problem (22) in the pulsating variables $\xi$,$\eta$,$\zeta$ with the denotations (28), (29), can be written as

$$(\xi - \xi_1)^2 + \eta^2 + \zeta^2 = \left(\frac{\nu s}{1-s}\right)^{2/3}\left[(\xi - \xi_3)^2 + \eta^2 + \zeta^2\right], \quad \xi_1 = s, \quad \xi_3 = -(1-s). \tag{30}$$

Let us denote that the three differential equations (25)–(27) and one algebraic equation (30) consist four variables $\xi$,$\eta$,$\zeta$ и $s$ that is why the system is closed.

The differential equations of motion (25)–(27) of the triangular restricted three-body problem in general case corresponding to the parameter $s = s(t) \neq const$ ($k = k(t) \neq const$) in the special non-inertial central reference frame in the pulsating variables are convenient for establishing exact particular analytical solutions. The mass parameter $\nu$ can be included into these equations in accordance to (28), (29).

While studying the solutions of differential equations of motion of the triangular restricted three-body problem in the special non-inertial central reference frame (25)–(27) and (30), it is convenient to distinguish the three possible cases:

$$1. \quad k = m_1/m_3 = \nu = const \tag{31}$$

$$2. \quad k = const \neq \nu = m_1/m_3 = const \tag{32}$$

$$3. \quad k = k(t) \neq const. \tag{33}$$

In each case it is needed to define the required four scalar values, uniquely satisfying the system of equations (25)–(27) and (30). Let us consider each case.

*5 The first case - the isosceles non-circular restricted three-body problem.*

Let consider particular and important case of the triangular restricted three-body problem (31), when the equations (25)–(27), (30) can get significantly simplified. Let the following condition take a place

$$k = m_1/m_3 = \nu = const > 0. \tag{34}$$

Let us note that in the case (34), the values of masses $m_1$ и $m_3$ are completely different. In this case, from (30)

$$\Delta_{21} = \Delta_{23} = \Delta. \tag{35}$$

From the equation (10) it is seen that, at $k = m_1/m_3$ the special non-inertial central reference frame $Gxyz$ transforms into the barycentric reference frame $G_0xyz$. It is well-known that the barycentric reference frame is inertial reference frame. At that, radius-vector of the barycenter is defined by the relation (10) in the absolute reference frame. Accordingly taking into account (35), the vector form of the forces center invariant to be transformed into the invariant of masses center

$$m_1\vec{r}_1 + m_2\vec{r}_2 = 0.$$

In this case from (35) it comes that the triangle formed by three bodies is isosceles during all time of motion and at vertex of this triangle is massless body. This case is studied by us in the works [22–24], but from different point of view. The isosceles restricted three-body problem is described in general case and it can be elliptical (circular in particular), parabolic, hyperbolic and rectilinear isosceles restricted three-body problem. Let us consider the most interesting case, when the following conditions can take a place in the equations (25)–(27), (30)

$$e \neq 0, \quad \zeta = 0, \quad k = m_1/m_3 = \nu = const > 0. \tag{36}$$

In this particular case we have the planar isosceles non-circular restricted three-body problem and some variables in the differential equations (25)–(27) will get simply

$$\sigma_2^2 = \sigma^2 = \frac{m_1 m_3}{(m_1 + m_3)^2}, \quad A = 1, \quad B = 0.$$

In the barycentric rotating reference frame in pulsating variables, the differential equations of motion of the planar isosceles non-circular ($e \neq 0$) restricted three-body problem is

$$\xi'' - 2\eta' - \frac{1}{1 + e\cos\theta}\left(1 - \frac{1}{(\rho^2 + \sigma^2)^{3/2}}\right)\xi = 0\,, \tag{37}$$

$$\eta'' + 2\xi' - \frac{1}{1 + e\cos\theta}\left(1 - \frac{1}{(\rho^2 + \sigma^2)^{3/2}}\right)\eta = 0. \tag{38}$$

Taking into account (36) and (28)–(29), from forces center invariant (30) expressed in pulsating variables, one can obtain

$$\xi = \xi^* = \frac{m_3 - m_1}{2\,(m_1 + m_3)} = const \neq 0. \tag{39}$$

Thus in equations of the planar isosceles non-circular ($e \neq 0$) restricted three-body problem (37)–(38) the axe $\xi$ is defined. This is constant value and defined by the formula (39). Taking into account (39), the equations of motion (37)–(38) will get more simply and we obtain dynamical system with one degree of freedom

$$\eta' = -\frac{\xi^*}{2\,(1 + e\cos\theta)}\left(1 - \frac{1}{(\eta^2 + 1/4)^{3/2}}\right), \tag{40}$$

$$\eta'' = \frac{\eta}{1 + e\cos\theta}\left(1 - \frac{1}{(\eta^2 + 1/4)^{3/2}}\right) \tag{41}$$

From the differential equations system (40), (41), one can obtain integrals identifying new trajectory in the planar non-circular ($e \neq 0$) isosceles restricted three-body problem

$$\xi^*\eta' + \eta^2 = c_1, \quad c_1 = \xi^*\eta_0' + \eta_0^2 = const, \quad \xi^* \neq 0. \tag{42}$$

The differential equation (42), depending on the value $c_1$, has three types of solutions. Let us emphasize that the particular case of the equations system (40), (41) when

$$e = 0, \quad r_{31} = a = const, \quad c_{31} = const \neq 0 \tag{43}$$

is studied by us in details in the works [25, 26]. In these works, the case (43) is studied through different methods.

*6 The second case. Reduction to quadrature of solutions of the planar circular triangular restricted three-body problem.*

In the case (32)

$$k = const \neq m_1/m_3 > 0, \quad (k \neq \nu) \tag{44}$$

all three bodies form non-isosceles triangle during all time of motion

$$\xi'' - 2\eta' - \frac{1}{1+e\cos\theta}\left(1 - \frac{A}{(\rho^2+\sigma_2^2)^{3/2}}\right)\xi = B \quad , \quad B \neq 0\,, \tag{45}$$

$$\eta'' + 2\xi' - \frac{1}{1+e\cos\theta}\left(1 - \frac{A}{(\rho^2+\sigma_2^2)^{3/2}}\right)\eta = 0, \tag{46}$$

$$\zeta'' + \frac{1}{1+e\cos\theta}\left(e\cos\theta + \frac{A}{(\rho^2+\sigma_2^2)^{3/2}}\right)\zeta = 0, \tag{47}$$

with

$$\rho^2 = \xi^2 + \eta^2 + \zeta^2, \quad \nu = \frac{m_1}{m_3} = const > 0, \quad \sigma_2^2 = \frac{k}{(1+k)^2} = const > 0,$$

$$A = \frac{\left[1 + \nu^{2/3}k^{1/3}\right]^{3/2}}{(1+k)^{1/2}(1+\nu)} = const > 0, \quad B = \frac{k-\nu}{(k+1)(1+\nu)} = const \neq 0.$$

Taking into account (28), (29) and the condition (44) the forces center invariant (30) can be written as

$$(\xi - \xi_1)^2 + \eta^2 + \zeta^2 = (\nu/k)^{2/3}\left[(\xi-\xi_3)^2 + \eta^2 + \zeta^2\right], \tag{48}$$

$$\xi_1 = \frac{1}{1+k} = const, \quad \xi_3 = -\frac{k}{1+k} = const.$$

Based on the obtained equations, we can establish exact particular analytical solutions of the planar triangular circular restricted three-body problem. Let take a place the following condition in the equations (45)–(47)

$$e = 0, \quad \zeta = 0, \quad k = const \neq m_1/m_3 > 0. \tag{49}$$

Taking into account (49), from the center of forces invariant (48)

$$\eta^2 + \xi^2 = E_1\xi + E_0, \tag{50}$$

$$E_1 = \frac{2(1+\varepsilon\nu)}{(1-\varepsilon)(1+\nu)} = const, \quad E_0 = -\frac{(1-\varepsilon\nu^2)}{(1-\varepsilon)(1+\nu)^2} = const.$$

$$1 - \varepsilon = 1 - (\nu/k)^{2/3} = const \neq 0$$

.

From the equation of motion (45)-(47), the Jacobi integral can be derived

$$\frac{1}{2}(\xi'^2 + \eta'^2) - \frac{1}{2}(\xi^2 + \eta^2) - \frac{A}{(\rho^2+\sigma_2^2)^{1/2}} - B\xi = C = const. \tag{51}$$

The existing of the two equations (50) and (51), in the case (49) allows us to reduce to quadrature the solution of the problem.

The parameter $k$, in accordance to the inequality (44), is defined from the condition of defining of possible motion region.

### *7 The third case*

The general case (33) is most interesting and sophisticated, that is why it shall be investigated in another work.

### *8 Conclusion*

In this work, the triangular restricted three-body problem is investigated analytically in the special non-inertial central reference frame with the origin at the center of forces. The solutions of differential equations of the triangular restricted three-body problem is in the rotating special non-inertial central reference frame in dimensionless pulsating variables. New exact particular solutions have been obtained.

In the planar triangular non-circular restricted three-body problem ($e \neq 0$) there have been found out new exact particular solutions of differential equations of motion in the form of isosceles triangle with variable height for arbitrary values of masses. There have been obtained new exact particular analytical solutions of differential equations of motion of the planar triangular circular restricted three-body problem ($e = 0$) in the form of non-isosceles triangle at arbitrary values of masses of the primary bodies. The basis of this non-isosceles triangle is distance between the primary bodies, and the ratio of lateral sides is permanent. A massless body is on vertex of this triangle.

We plan to perform detailed analysis of the equations of motion and to investigate stability of obtained new solutions of the triangular restricted three-body problem. The obtained exact particular analytical solutions can be effectively used for topological analysis of the general solution.

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# Шектелген үшбұрышты үш дене мәселесінің жаңа нақты дербес аналитикалық шешімдері

Үшбұрышты шектелген үш дене мәселесі аналитикалық жолмен арнайы инерциалды емес централды санақ жүйесінде қарастырылған. Бұл санақ жүйенің басы күштер центрінде орналасады. Негізгі екі дене массалары кез-келген шама. Айналмалы санақ жүйесінің пульсирленген айнымалыларында үшбұрышты шектелген үш дене мәселесінің өлшемсіз дифференциалдық теңдеулері зерттелді. Шеңберлік емес жазық шектелген үш дене мәселесінде жаңа нақты дербес теңбүйірлі биіктігі айналмалы үшбұрыш түрінде аналитикалық шешімдер анықталды. Және теңбүйірлі емес үшбұрыш түріндегі жазық шеңберлік шектелген үш дене мәселесінің жаңа аналитикалық теңдеулер классы табылды. Теңбүйірлі емес үшбұрыштың негізін екі негізгі денелер арақашықтығы құрайды, теңбүйірлі емес үшбұрыштың бүйір қабырғаларының қатынасы тұрақты шама және осы теңбүйірлі емес үшбұрыштың төбесінде массасы шексіз аз дене орналасады. Табылған нақты дербес шешімдерді жалпы мәселені зерттеу үшін топологиялық талдауға қолдануға болады.

*Кілт сөздер:* шектелген үш дене мәселесі, инерциалды емес санақ жүйесі, күштер центрінің инварианты, нақты дербес аналитикалық шешімдер.

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## Новые точные частные аналитические решения треугольной ограниченной задачи трех тел

Аналитически исследована треугольная ограниченная задача трех тел в специальной неинерциальной центральной системе координат с началом в центре сил исследуемой задачи. При этом массы основных тел произвольные. Изучены решения безразмерных дифференциальных уравнений движения треугольной ограниченной задачи трех тел во вращающейся системе координат в пульсирующих переменных. В некруговой плоской ограниченной задаче трех тел установлены новые точные аналитические частные решения, в виде равнобедреннего треугольника переменной высоты. Также аналитически найден новый класс решений плоской круговой ограниченной задачи трех тел в виде неравнобедренного треугольника. Основанием неравнобедренного треугольника является расстояние между основными телами, отношение боковых сторон неравнобедренного треугольника постоянное, и на вершине этого неравнобедренного треугольника находится тело малой массы. Установленные точные частные аналитические решения можно эффективно использовать для топологического анализа общего решения проблемы.

*Ключевые слова:* ограниченная задача трех тел, неинерциальная система координат, инвариант центра сил, точные частные аналитические решения.

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# Movement of a material particle on an inclined plane all the points of which describe circles in oscillatory motion in the same plane

Differential equations of material particle movement on an inclined rough plane, which performs oscillatory motion in such a way that its every point describes a circle in the same plane, have been deduced. Peculiarities of relative particle movement on a plane depending on the angle of its inclination to the horizon have been investigated. The equations have been solved using numerical methods. Relative velocities have been found and particle motion trajectories have been constructed. Kinematic characteristics of relative particle movement depending on the angle of plane inclination, angular velocity, the coefficient of particle friction on a plane and the radius of circular motion of plane points have been determined.

*Keywords:* inclined plane, oscillatory motion, relative motion, particle, kinematic parameters.

## *Introduction*

An inclined plane is a general purpose construction element of numerous agricultural machines [1]. In the course of processing technological material moves on it. Particle movement on a horizontal plane that performs oscillatory straight-line or circulatory motion is the best investigated. As for an inclined plane, investigations are mainly conducted for its rectilinear reciprocating horizontal oscillations in the direction of plane inclination or in the transverse direction [1]. If there are non-rectilinear plane oscillations, when all its points describe a circle and the plane itself is inclined, the movement of technological material changes significantly.

In addition to the fundamental monograph [1] that covers rectilinear reciprocating oscillations, there are works focused on non-rectilinear plane oscillations. Academician P.M. Zaika [2] investigated the movement of a spherical particle on a horizontal plane, which performs translational oscillations in a circle and other [3, 7]. In fact, the problem of material particle movement on a plane that performs circular oscillatory motion was first solved in geometric interpretation [8], generalized and applied to the cases of elliptical vibrations by I.I. Blekhman [9]. Investigations of material particle movement on a rough horizontal plane, that performs horizontal translational oscillations on various curves, are covered in the papers [10–19].

## *Material and research methods*

Let us locate a plane in such a way that it is inclined to the horizon at an angle $\beta$ (Fig. 1). A particle performs relative movement on an inclined plane, where there are plane coordinates $ouv$ arranged in such a way that $ou$ axis in directed in the line of the greatest inclination. The inclined plane together with plane coordinates perform oscillations in such a way that all the points of the plane describe circles of radius $R$ in the same plane (Fig. 1,a, these circles are presented only in the apexes of the rectangle, which limits the plane). Absolute particle motion is considered relative to the fixed coordinate system $Oxyz$, where $Oy$ axis coincides with $ov$ axis, and there is the angle $\beta$ between the inclined plane $ouv$ and the coordinate horizontal plane $Oxy$. The origin of the moving coordinate system (point $o$) describes a circle as well. Fig. 1 presents the two systems at the time when their coordinate origins coincide.

In order to develop differential equations of particle movement, it is necessary to use its absolute trajectory in the fixed coordinates $Oxyz$. The absolute trajectory of a particle is written as the sum of the corresponding components in the translational motion and in the relative motion:

$$x = x_t + x_r; \qquad y = y_t + y_r; \qquad z = z_t + z_r, \tag{1}$$

where $x_t = x_t(t)$; $y_t = y_t(t)$; $z_t = z_t(t)$ is a trajectory of translational motion as a function of time;
$x_r = x_r(t)$; $y_r = y_r(t)$; $z_r = z_r(t)$ is a trajectory of relative motion as a fuction of time.

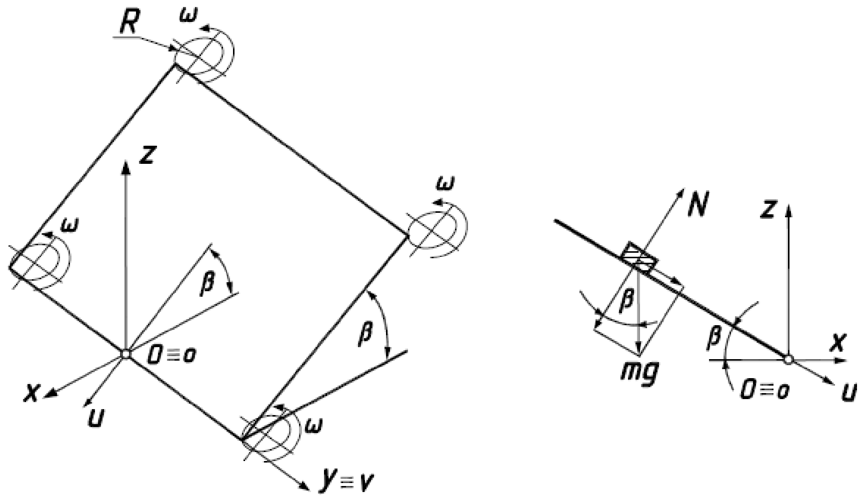


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Figure 1. Consideration of particle movement on an inclined plane with all its points describing circles at oscillations in the same plane: a) mutual arrangement of the moving coordinate system *ouv* and the fixed coordinate system *Oxyz* at the initial moment, when their coordinate origins coincide; b) particle position on a plane, when it is projected into a line

Every point of an inclined plane, including the origin of the moving coordinate system *ouv*, describes a circle of radius $R$. In the projections on the axes of the fixed coordinate system, relative motion of a plane is presented by the following parametric equations:

$$x_t = R\cos\beta\cos\omega t; \qquad y_t = R\sin\omega t; \qquad z_t = -R\sin\beta\cos\omega t, \tag{2}$$

where $\omega$ is angular velocity of rotation of every point of a plane.

A particle slides on an inclined plane and its sliding trajectory in the moving coordinate system *ouv* is written as a function of time $t$: $u = u(t)$; $v = v(t)$. In the projections on the axes of the fixed coordinate system, relative particle movement is described by the following parametric equations:

$$x_r = u\cos\beta; \qquad y_r = v; \qquad z_r = -u\sin\beta. \tag{3}$$

By summing translational and relative motion applying the formula (1), we obtain:

$$x = R\cos\beta\cos\omega t + u\cos\beta; \qquad y = R\sin\omega t + v; \qquad z = -R\sin\beta\cos\omega t - u\sin\beta. \tag{4}$$

The dependences $u = u(t)$; $v = v(t)$, that describe the trajectory of relative motion (particle sliding on an inclined plane), are the unknown functions that must be determined. By differentiating the equations (4) with respect to the time $t$, the projections of absolute particle velocity on the fixed coordinate system *Oxyz* are obtained:

$$\begin{aligned} x' &= -R\omega\cos\beta\sin\omega t + u'\cos\beta\,; \\ y' &= R\omega\cos\omega t + v'\,; \\ z' &= R\omega\sin\omega t - u\sin\beta. \end{aligned} \tag{5}$$

Differentiation of the expressions (5) allows for projecting absolute acceleration:

$$\begin{aligned} x'' &= -R\omega^2\cos\beta\cos\omega t + u''\cos\beta\,; \\ y'' &= -R\omega^2\sin\omega t + v''\,; \\ z'' &= R\omega^2\sin\beta\cos\omega t - u''\sin\beta. \end{aligned} \tag{6}$$

Let us deduce a motion equation in the form of $m\overline{w} = \overline{F}$, where $m$ is particle mass, $\overline{w}$ is absolute acceleration vector, $\overline{F}$ is the resultant vector of the forces applied to a particle. Such forces include weight force $mg$ ($g = 9,81 m/s^2$), the response $N$ of an inclined plane and friction force $fN$ at particle sliding on a plane ($f$ is friction coefficient). All the forces must be projected onto the axes of the fixed coordinate system.

Weight force is directed downwards, thus, its projections are written as:

$$\{0; \qquad 0; \qquad -mg\}\,.$$

The response $N$ of a plane is perpendicular to it (Fig. 1,b) and has the following projections:

$$\{N\sin\beta\,; \qquad 0\,; \qquad N\cos\beta\}\ .$$

Since friction force is directed at a tangent to the trajectory of relative particle motion in the opposite direction, let us find the projections of the tangent vector. They are determined by the first derivatives of the equations (3):

$$x_r = u'\cos\beta; \qquad y_r = v'; \qquad z_r = -u'\sin\beta. \tag{7}$$

The geometric sum of the components (7) provides the velocity of particle sliding on the surface of a cylinder in relative motion:

$$V_r = \sqrt{x_r'^{\,2} + y_r'^{\,2} + z_r'^{\,2}} = \sqrt{u'^2 + v'^2}. \tag{8}$$

The unit tangent vector in the projections on the axes of the moving coordinate system $Oxyz$ is obtained from dividing the projections (7) by the vector value (8):

$$\left\{\frac{u'\cos\beta}{\sqrt{u'^2+v'^2}}; \qquad \frac{v'}{\sqrt{u'^2+v'^2}}; \qquad -\frac{u'\sin\beta}{\sqrt{u'^2+v'^2}}\right\}. \tag{9}$$

Let us break down the vector equation $m\overline{w} = \overline{F}$ in the projections on the axes of the fixed coordinate system, taking into account that the friction force $fN$ is directed along the unit vector (9) oppositely to it:

$$\begin{aligned} mx'' &= N\sin\beta - fN\tfrac{u'\cos\beta}{\sqrt{u'^2+v'^2}}\,; \\ my'' &= -fN\tfrac{v'}{\sqrt{u'^2+v'^2}}; \\ mz'' &= -mg + N\cos\beta + fN\tfrac{u'\sin\beta}{\sqrt{u'^2+v'^2}}\,. \end{aligned} \tag{10}$$

Let us insert other derivatives (projections of absolute acceleration) from (6) into the equation (10) and we obtain the system of three equations:

$$\begin{aligned} m\left(-R\omega^2\cos\beta\cos\omega t + u''\cos\beta\right) &= N\sin\beta - fN\tfrac{u'\cos\beta}{\sqrt{u'^2+v'^2}}\,; \\ m\left(-R\omega^2\sin\omega t + v''\right) &= -fN\tfrac{v'}{\sqrt{u'^2+v'^2}}; \\ m\left(R\omega^2\sin\beta\cos\omega t - u''\sin\beta\right) &= -mg + N\cos\beta + fN\tfrac{u'\sin\beta}{\sqrt{u'^2+v'^2}}. \end{aligned} \tag{11}$$

The system (11) includes three unknown functions: $N = N(t)$, $u = u(t)$ and $v = v(t)$. By solving it with respect to $N$, $u''$ and $v''$, we obtain a very simple expression for $N$:

$$N = mg\cos\beta\,. \tag{12}$$

It follows from (12) that the force of $N$ surface pressure on a particle is a steady-state one. It is possible to obtain tentative verification that the mass $m$ in the equations reduces, if it is substituted in (12) and (10). After the rearrangement, the dependences $u''$ and $v''$ take the following forms:

$$\begin{aligned} u'' &= R\omega^2\cos\omega t + g\sin\beta - fg\tfrac{u'\cos\beta}{\sqrt{u'^2+v'^2}}\,; \\ v'' &= R\omega^2\sin\omega t - fg\tfrac{v'\cos\beta}{\sqrt{u'^2+v'^2}}\,. \end{aligned} \tag{13}$$

The system (13) cannot be integrated in the analytical form. It must be solved using numerical methods. Analytical solution can be obtained for a special case when $f = 0$, that is for an absolutely smooth plane:

$$\begin{aligned} u &= \tfrac{gt^2}{2}\sin\beta + c_1 t - R\cos\omega t\,; \\ v &= c_2 t - R\sin\omega t\ , \end{aligned} \tag{14}$$

where $c_1$, $c_2$ is integration constants.

In order to find the absolute trajectory of a particle, it is necessary to insert the expressions (14) into the parametric equations (4):

$$\begin{aligned} x &= \tfrac{gt^2}{2}\sin\beta\cos\beta + c_1 t\cos\beta\,; \\ y &= c_2 t\,; \\ z &= -\tfrac{gt^2}{2}\sin^2\beta - c_1 t\sin\beta\,. \end{aligned} \tag{15}$$

The equations (15) describe a parabola that is located in an inclined plane.

*Results*

The research is conducted for a case, when $\beta = 0$, that is to say, in the case of a horizontal plane, the absolute trajectory transforms into a straight line. This is predictable, since, if there is no friction, a particle does not response to plane oscillations moves in absolute motion as if it is fixed. The relative trajectory, which is the result of particle sliding on a plane, takes the correspondent curvilinear form near the absolute trajectory. If $\beta = 0$ and $f \neq 0$, numerical integration of the equations (13) shows that the trajectory of particle relative motion is a circle. A particle slides in it after its motion is stabilized and it is possible to find the analytical solution of the differential equation system [10-12] for this case. The form of the relative trajectory during the transient period, that is after a particle gets onto a plane and up to its motion stabilization, depends on the reference conditions of integration: the value of velocity and its direction at the moment of getting onto a plane.

Let us consider that a particle falls vertically and meets the plane at a right angle. Let us assume that at the moment of meeting a plane its absolute velocity is equal to zero. Since a plane performs oscillatory motion at this moment, there is particle sliding on a plane. The value and the direction of sliding velocity (that is to say, the velocity of relative motion) is equal to the analogical values of the translation motion of a plane at the point of particle entering but the velocity is oppositely directed. The point of particle entering depends on the time $t_0$. Since every plane point describes a circle of radius $R$, a particle enters a certain point of this circle, which is determined by the radius vector angle of rotation about the angle $\varphi_0 = \omega t_0$. Having inserted this value into the equation (2), it is possible to determine the point of particle entering a plane in the fixed coordinate system. The velocity value is determined by differentiating the equations (2). For example, $y'_t = R\omega\cos\omega t = R\omega\cos\omega t_0 = R\omega\cos\varphi_0$. Thus, $v'(\varphi_0) = -y'_t = -R\omega\cos\varphi_0$. Similarly, $u'(\varphi_0) = -x'_t = R\omega\sin\varphi_0$. These data are the reference conditions of integration. Fig. 2,a presents the trajectories of relative particle motion when it enters a plane after $45^0$ rotation of this plane around a circle in translational motion. Fig. 2,b presents the graph of change in sliding velocity, which is determined from the formula (8). Thus, the trajectory of relative particle motion after its motion is stabilized is a circle and, after this, the relative velocity becomes constant. The paper [7–9] provides the dependence of the circle radius $\rho_r$ is the trajectory of relative particle motion after its motion is stabilized – on $R$, $f$ and $\omega$:

$$\rho_r = R\sqrt{1 - \left(\frac{fg}{R\omega^2}\right)^2}. \qquad (16)$$

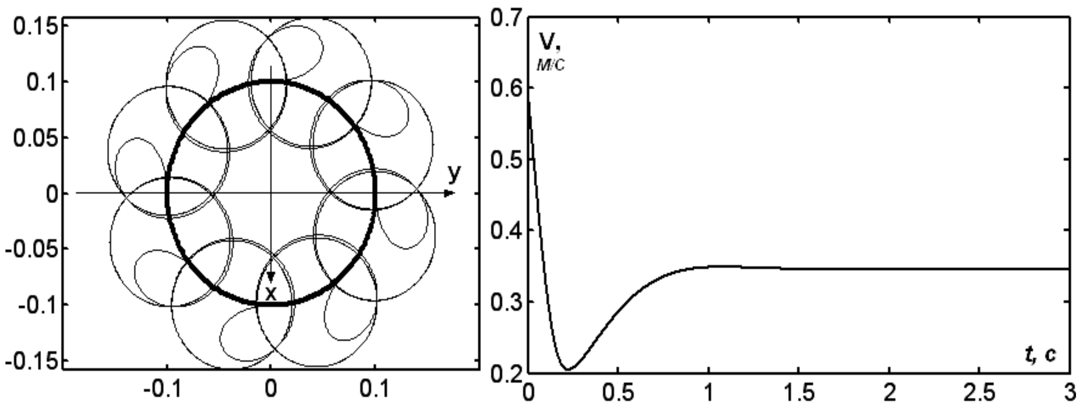


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Figure 2. Kinematic characteristics of relative motion at *ω=6* $s^{-1}$, *R=0,1 m*, *f=0,3*: a) relative trajectories originate from the circle of plane translational motion after $45^0$; b) graph of relative velocity $V_r$ change

The (16) shows that at the set values of $R$ and *f*, there is a critical value of the angular velocity $\omega$, at which relative motion is possible. If the angular velocity of plane oscillation is lower than the critical one, sliding is not possible: a particle «sticks» on a plane. If the angular velocity $\omega$ increases, the kinematic characteristics of a particle change: the radius $\rho_в$ of a circle of relative motion increases and reaches the one of a translational motion circle (Fig. 3,a) and the time of relative velocity stabilization increases (Fig. 3,b).

Let us determine the patterns of particle movement on an inclined plane that oscillates. The investigations show that, if there is a plane inclination beginning from the horizontal position, the trajectories of relative motion transform from circles into curves that are similar to cycloids (extended, regular, curtailed), here, their transformations take place with respect to the inclination of a plane in the order enumerated in the parentheses.

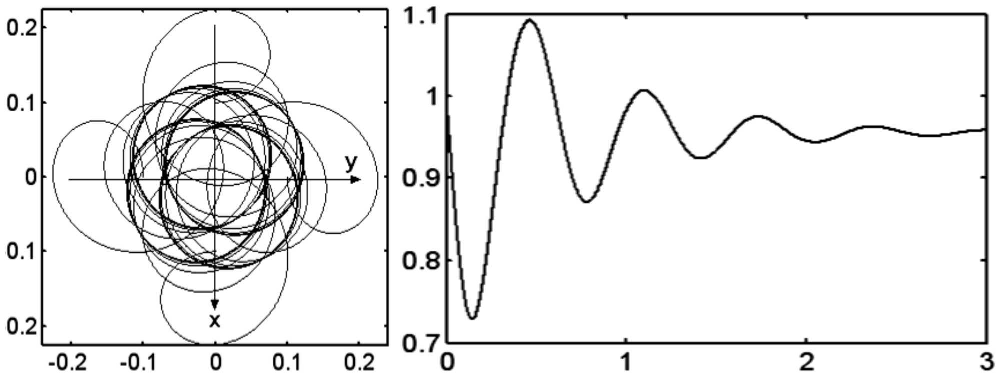


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Figure 3. Kinematic characteristics of relative motion at $\omega = 10s^{-1}$, *R=0,1 m*, *f=0,3*:
a) relative trajectories originate from the circle of plane translational motion after $90^0$;
b) graph of relative velocity $V_r$ change

Fig. 4 presents the correspondent graphs of trajectories and velocities for $\omega = 6s^{-1}$ and $\omega = 10s^{-1}$ at the plane inclination being $\beta = 2^0$. Similar to a horizontal plane, with the increase of the angular velocity $\omega$, the value of relative oscillations increases, here, their propagation direction does not coincide with the line of the greatest inclination, however, with the increase of the angular velocity it increasingly reaches it. As for the initial conditions, if there are high angles of plane inclination, it is necessary to take into account the velocity of particle motion in the vertical direction downwards at the moment of its entering a plane ($V_0$). Relative sliding velocity is increased by the component $V_0\sin\beta$, that is $u'(\varphi_0) = R\omega\sin\varphi_0 - V_0\sin\beta$. This component plays its role only at the beginning of movement. Fig. 4 presents the graphs after motion stabilization. Particle relative velocity changes similar to a sinusoid, here, its maximum and minimum values remain constant. It is obvious that the oscillatory motion of a particle in the direction close to the line of the greatest inclination takes place uniformly, that is, the propagation velocity of sliding is constant. If the angle of plane inclination increases, for example, to $\beta = 20^0$, the pattern of oscillations changes (Fig. 5). The trajectory becomes similar to a curtailed cycloid with a pitch that increases (Fig. 5,a) and relative velocity at the equal amplitude changes in such a way that its extremum values increase in linear fashion (Fig. 5,b). It means that oscillations have accelerated propagation.

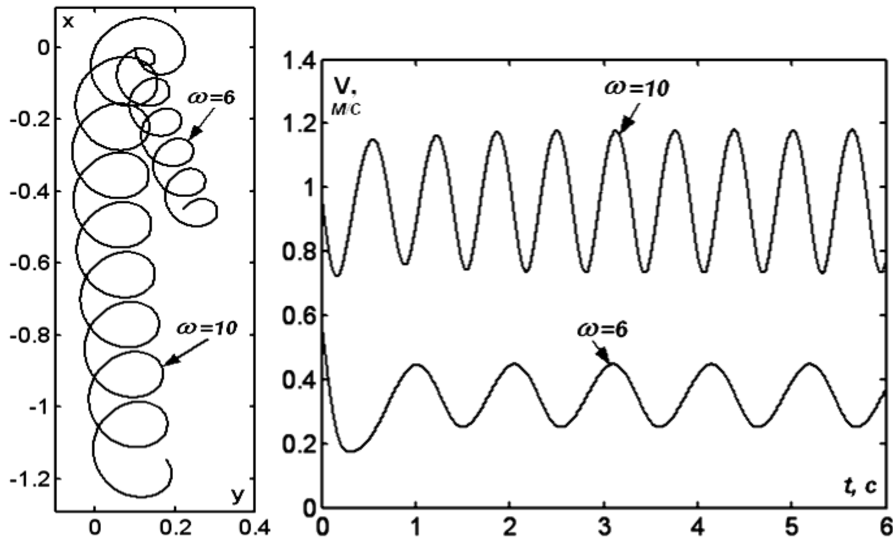


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Figure 4. Kinematic characteristics of relative motion at
$\beta = 2^0$, *R=0,1*, *f=0,3*:
a) relative trajectories; b) graphs of relative velocities

There is a question: at what value of the angle $\beta$ the pattern of propagation of oscillations transforms from the uniform to the accelerated one. It can be assumed that such a limit is the angle $\beta$, which is equal to a friction angle, that is $\beta$=Arctg$f$ (at $f$=0,3 $\beta = 16,7^0$). However, it is not the case, since at such an angle oscillations have accelerated propagation. It is obvious that the angle $\beta$ is smaller than the friction angle.

It was determined by trial and error method: $\beta = 15,7^0$. In this case the trajectory is a curve, which is similar to a curtailed cycloid (Fig. 6,a). Relative velocity is stabilized in such a way that its value is changed within the limits of *1. . . 3 m/s* (Fig. 6,b). The investigations show that the limit between uniform particle oscillations and accelerated oscillations is the angle of plane inclination, which is somewhat smaller than the friction angle.

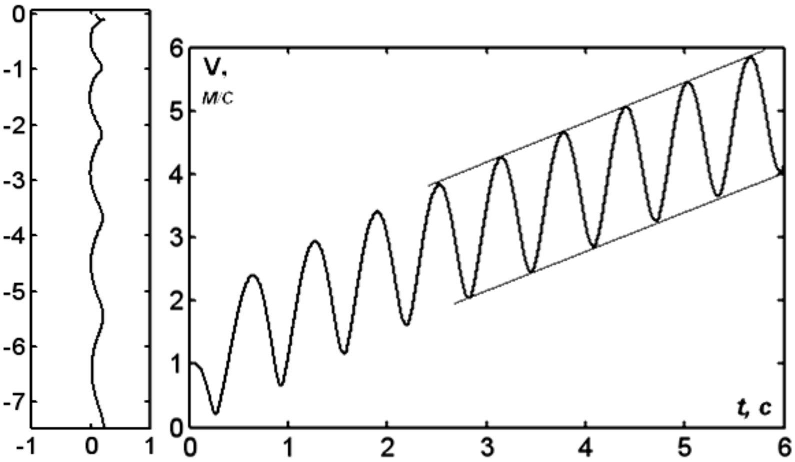


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Figure 5. Kinematic characteristics of relative motion at
$\beta = 20^0$, *R=0,1*, $\omega = 10s^{-1}$, *f=0,3*:
a) relative trajectory; b) graph of relative velocity change

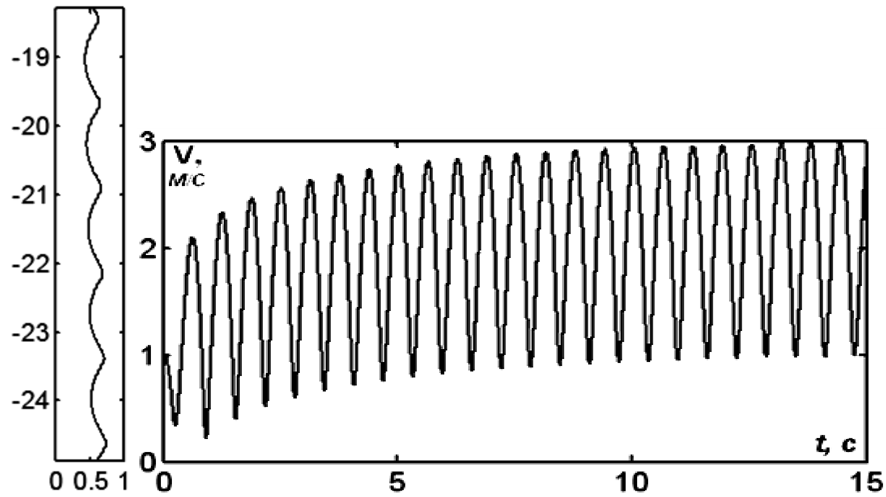


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Figure 6. Kinematic characteristics of relative motion at
$\beta = 15,7^0$, *R=0,1*, $\omega = 10s^{-1}$, *f=0,3*:
a) relative trajectory; b) graph of relative velocity change

According to Fig. 4 and 6, with the increase of the angle of plane inclination, the trajectory of a particle changes its form: it transforms from an extended cycloid into a curtailed one. It is logical to assume that at a certain intermediate angle $\beta$ it may be a regular cycloid. Such an intermediate angle was determined by trial and error method as well: $\beta = 11^0$. The characteristic feature of such oscillations is the fact that a particle drastically changes is movement direction at the points of trajectory winding (Fig. 7,a), which is not possible without stopping. According to the graph of regular velocity change, its value ranges within the limits of *0. . . 2 m/s* (Fig. 7,b), that is, at the point of winding the velocity is equal to zero.

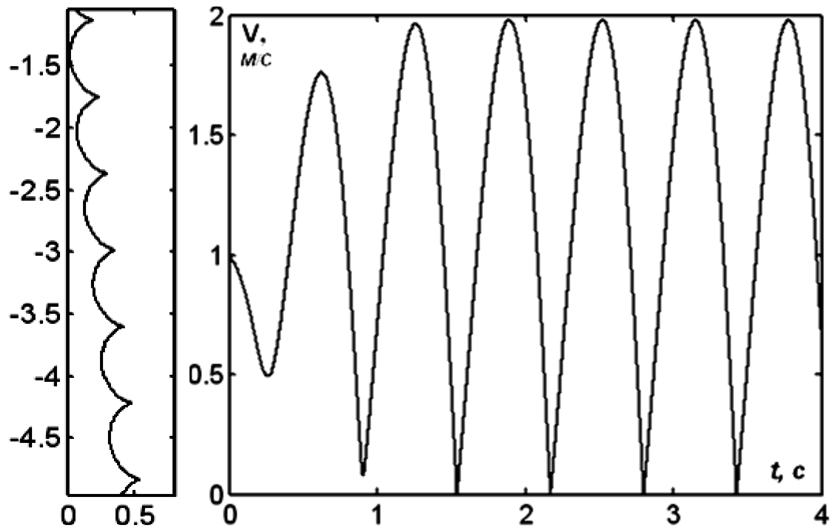


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Figure 7. Kinematic characteristics of relative motion at $\beta = 11^0$, *R=0,1,* $\omega = 10s^{-1}$, *f=0,3*:
a) relative trajectory; b) graph of relative velocity change

*Conclusions*

The patterns of particle relative motion on a rough inclined plane, all the point of which describe circles in oscillatory motion in the same plane have been determined. At the inclination angle being $\beta = 0^0$, that is, in the case of a horizontal plane, a particle describes a circle in relative motion, when the minimum angular velocity of plane oscillations is reached. If there is an increase of the angular velocity, a circle radius is the trajectory of relative motion is increases approaching to the radius of the circle of translational motion of plane oscillations. If the plane is inclined, beginning from a horizontal position, the trajectories of relative motion transform from circles into curves, which are similar to cycloids (extended, regular, curtailed), here, their transformations take place with respect to plane inclination in the order enumerated in the parentheses. With the increase of the angular velocity $\omega$, the pitch and the amplitude of relative particle oscillations increase, here, their propagation direction does not coincide with the line of the greatest inclination, however, with the increase of the angular velocity it approaches to it more and more. Until the moment when there is the boundary value of the inclination angle $\beta$ reached, which is somewhat smaller than the friction angle, oscillatory particle movement in the direction close to the line of the greatest inclination takes place uniformly, that is to say, the propagation velocity of oscillations in constant. Relative particle velocity changes similar to a sinusoidal law, here, its maximum and minimum values remain constant. If there is further increase of the inclination angle $\beta$, the trajectory pitch becomes changeable, that is to say, it increases and relative velocity changes at the same amplitude in such a way that its extremum values increase according to the linear law, that is to say, oscillations have accelerated propagation.

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# Барлық нүктелері көлбеу жазықтықта тербелмелі қозғалыста дөңгелек салатын материалдық бөлшектің қозғалысы

Көлбеу бүртік жазықтық бойымен қозғалған материалдық бөлшектің осы жазықтықта тербелмелі қозғалыста әрбір нүктесі дөңгелек сызатындай қозғалысының дифференциалдық теңдеуі құрылған. Жазықтық бойымен көкжиекке көлбеу бұрышына тәуелді бөлшектің қозғалысының салыстырмалы ерекшеліктері зерттелді. Теңдеулер сандық әдістермен шешілді. Салыстырмалы жылдамдықтар табылды және бөлшектер қозғалысының траекториялары салынды. Жазықтыққа көлбеу бұрышына, бұрыштық жылдамдыққа, жазықтық бойымен бөлшектің үйкелу коэффициентіне, жазықтық нүктелерінің қозғалысы салған дөңгелек радиусына байланысты бөлшектердің салыстырмалы қозғалысының кинематикалық характеристикалары тағайындалды.

*Кілт сөздер:* көлбеу жазықтық, тербелмелі қозғалыс, салыстырмалы қозғалыс, бөлшек, кинематикалық параметрлер.

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## Движение материальной частицы по наклонной плоскости, все точки которой в колебательном движении описывают круги в этой же плоскости

Составлены дифференциальные уравнения движения материальной частицы по наклонной шероховатой плоскости, осуществляющей колебательное движение таким образом, что каждая ее точка описывает круг в этой же плоскости. Исследованы особенности относительного движения частицы по плоскости в зависимости от ее угла наклона к горизонту. Уравнения решены численными методами. Найдено относительное скорости и построена траектория движения частиц. Установлены кинематические характеристики относительного движения частицы в зависимости от угла наклона плоскости, угловой скорости, коэффициента трения частицы по плоскости и радиуса круга движения точек плоскости.

*Ключевые слова:* наклонная плоскость, колебательное движение, относительное движение, частица, кинематические параметры.

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# ЕСКЕ АЛУ
# ПАМЯТИ УЧЕНОГО
# IN MEMORIAM OF SCIENTIST

## Жизнь, посвященная науке

Ушел из жизни доктор физико-математических наук, профессор Дулат Сыздыкбекович ДЖУМАБАЕВ (1954–2020). Дулат Сыздыкбекович прошел свой жизненный путь стремительно и ярко, оставив в математике светлый след. Он был высококвалифицированным специалистом в области качественной теории дифференциальных уравнений. Был замечательным педагогом, не только много преподавал, но и ежедневно проводил многочасовые консультации, учил своих многочисленных учеников профессионализму, честности и терпению в работе, самоотверженности в научном труде.

Д.С. Джумабаев родился 11 апреля 1954 года в пос. Кантаги Туркестанского района Южно-Казахстанской области. С 1961 по 1971 гг. учился в СШ № 386 г. Туркестана. В 1971 г. поступил на механико-математический факультет Казахского государственного университета им. С.М. Кирова. В 1976 г. с отличием окончил вуз и поступил в аспирантуру Института математики и механики АН КазССР. После успешного завершения аспирантуры в 1979 г. был принят на работу в лабораторию, возглавляемую академиком О.А. Жаутыковым.

Начав с должности младшего научного сотрудника лаборатории обыкновенных дифференциальных уравнений, он прошел свой научный и трудовой путь до заведующего одного из ведущих подразделений Института математики — лаборатории дифференциальных уравнений (с 1996 г.). В 1980 г. защитил диссертацию на тему «Краевые задачи с параметром для обыкновенных дифференциальных уравнений в банаховом пространстве» на степень кандидата физико-математических наук (по специальность 01.01.02 – «Дифференциальные уравнения»).

В 1998 г. Д.С. Джумабаеву присуждено звание профессора (специальность 01.01.00 – «Математика»). Под его руководством были защищены две докторские и более 20 кандидатских диссертаций. Он являлся научным руководителем 5 докторантов.

В 2004–2005 гг. Д.С. Джумабаев — председатель Экспертной комиссии по математике и информатике Комитета по надзору и аттестации в сфере образования и науки МОН РК.

Им опубликовано более 300 работ в авторитетных периодических изданиях, таких как «Journal of Computational and Applied Mathematics», «Journal of Mathematical Analysis and Applications», «Mathematical Methods in Applied Sciences», «Computational Mathematics and Mathematical Physics», «Journal of Mathematical Sciences», «Ukranian Mathematical Journal» и др. Результаты исследований апробированы на многих международных симпозиумах и конференциях. Его научные результаты получили широкое признание в республике и за рубежом у специалистов в области дифференциальных уравнений и вычислительной математики. Сформированное им научное направление получило дальнейшее развитие в работах его учеников, которые успешно работают в ведущих университетах нашей страны.

До сегодняшнего дня он являлся руководителем научного проекта по грантовому финансированию, выполняемого в Институте математики и математического моделирования, руководителем научного семинара по качественной теории дифференциальных уравнений научным экспертом Государственной экспертизы МОН РК, членом Диссертационного совета Д 53.04.01 при Институте математики, председателем секции математики Ученого совета ИМ, членом редакционной коллегии журнала «Вестник Карагандинского университета. Серия математика», рецензентом журналов «Известия НАН РК. Серия физико-математическая», «Математический журнал», «Вестник Казахского национального технического университета. Серия Математика, информатика».

За многолетнюю плодотворную научную деятельность Д.С. Джумабаев награжден нагрудным знаком МОН РК «За вклад в развитие науки и техники» (2005), Почетной грамотой МОН РК. Был удостоен звания «Лучший преподаватель вуза — 2019», которое так и не успел получить...

Дулат Сыздыкбекович спешил жить, преобразовывать мир вокруг себя, жить от души. Осталась память, вобравшая в себя яркий научный талант и человеческое обаяние Дулата Сыздыкбековича, остался его творческий заряд, обязывающий знавших его коллег и учеников продолжать его дело.

Выражаем искренние соболезнования родным и близким Дулата Сыздыкбековича. Светлая ему память!

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